

ON ZERMELO'S THEOREM

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ABSTRACT. A famous result in game theory known as Zermelo's theorem says that "in chess either White can force a win, or Black can force a win, or both sides can force at least a draw". The present paper extends this result to the class of all finite-stage two-player games of complete information with alternating moves. It is shown that in any such game either the first player has a winning strategy, or the second player has a winning strategy, or both have unbeatable strategies.

1. In this note we generalize the following proposition usually referred to in the modern game-theoretic literature as Zermelo's theorem (see the paper by Zermelo [9] and its discussion and further references in Schwalbe and Walker [8]).

Theorem 1. *In chess either White has a winning strategy, or Black has a winning strategy, or both have strategies guaranteeing at least a draw.*

The main result of this paper is as follows.

Theorem 2. *In any finite-stage two-player game with alternating moves, either (i) player 1 has a winning strategy, or (ii) player 2 has a winning strategy, or (iii) both have unbeatable strategies.*

We emphasize that in this theorem we speak of *any* finite-stage two-player game with alternating moves, with *any* action sets and *any* real-valued payoff functions, so that the formulation of the result has a maximum level of generality.¹

The notions of unbeatable and winning strategies are defined in the general context as follows. Consider a game of two players who select strategies ξ and η from some sets and get payoffs $U(\xi, \eta)$ and $V(\xi, \eta)$. We call a strategy ξ of player 1 *unbeatable* (resp. *winning*) if $U(\xi, \eta) \geq V(\xi, \eta)$ (resp. $U(\xi, \eta) > V(\xi, \eta)$) for any η . Unbeatable and winning strategies η of player 2 are defined analogously in terms of the inequalities $U(\xi, \eta) \leq V(\xi, \eta)$ and $U(\xi, \eta) < V(\xi, \eta)$ holding for each ξ . The idea of these concepts goes back to the seminal works of Bouton [4] and Borel [3]. In

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¹In the usual treatment of Zermelo's theorem, one assumes the finiteness of the players' action spaces (see, e.g., Aumann [2]).

modern contexts (mathematical finance and evolutionary game theory), they were developed in Amir et al. [1] and Kojima [7].

The notions introduced are regarded in this work as primitive solution concepts for strategic games. We do not try to reduce them to standard ones: saddle point, Nash equilibrium, dominant strategy or their versions. This approach, contrasting with the common game-theoretic methodology, constitutes the main element of novelty in this paper. It enables us to formulate the above result in its most general and natural form and thus to put a full stop to the chain of controversies around Zermelo's theorem (see [8]) that lasted for more than a century.

2. In a *finite-stage game with alternating moves*, players 1 and 2 make moves (take actions) sequentially by selecting elements from two given *action sets* A and B . At stage 0 player 1 makes a move a_0 ; then player 2, having observed the player 1's move a_0 , makes a move $b_0 = y_0(a_0)$. At stage 1 player 1 makes a move $a_1 = x_1(a_0, b_0)$ depending on the previous moves a_0 and b_0 ; then player 2 makes a move $b_1 = y_1(a_0, b_0, a_1)$, and so on. At stage t ($t \leq N$) player 1 makes a move $a_t = x_t(a^{t-1}, b^{t-1})$ depending on the sequences of the previous actions

$$a^{t-1} = (a_0, a_1, a_2, \dots, a_{t-1}) \text{ and } b^{t-1} = (b_0, b_1, b_2, \dots, b_{t-1})$$

up to time $t-1$, and then player 2 makes a move $y_t(a^t, b^{t-1})$ depending on $a^t = (a_0, a_1, a_2, \dots, a_t)$ and $b^{t-1} = (b_0, b_1, b_2, \dots, b_{t-1})$. The game terminates at stage N , where N is some given natural number.

A *strategy of player 1* is a sequence

$$\xi = \{x_0, x_1(a^0, b^0), x_2(a^1, b^1), x_3(a^2, b^2), \dots, x_N(a^{N-1}, b^{N-1})\}$$

where x_0 is the initial action of player 1 and $x_t(a^{t-1}, b^{t-1})$ ($1 \leq t \leq N$) is a function specifying what move $a_t = x_t(a^{t-1}, b^{t-1})$ should be made at stage t given the history (a^{t-1}, b^{t-1}) of the previous moves of the players. To specify a *strategy of player 2* one has to specify a sequence of functions

$$\eta = \{y_0(a^0), y_1(a^1, b^0), y_2(a^2, b^1), \dots, y_N(a^N, b^{N-1})\}$$

indicating what move $b_t = y_t(a^t, b^{t-1})$ should be made at stage t given the history (a^t, b^{t-1}) of the previous moves of the players.

The *outcome of the game* $h(\xi, \eta)$ resulting from the application of the strategies ξ and η is described by the whole history of play

$$h(\xi, \eta) = (a^N, b^N) = (a_0, a_1, a_2, \dots, a_N, b_0, b_1, b_2, \dots, b_N).$$

Once the outcome $h(\xi, \eta)$ of the game is known, the players get their payoffs $U(\xi, \eta) = u(h(\xi, \eta))$ and $V(\xi, \eta) = v(h(\xi, \eta))$, where $u(h)$ and $v(h)$ are the given *payoff functions* defined for all histories h .

In the course of the proof of Theorem 2, we will establish the existence of winning and unbeatable strategies having a special structure: basic strategies. We call a strategy *basic* if moves of the player using this strategy depend only on the previous moves of the rival (there is no need to memorize your own moves). Thus a basic strategy of player 1 is a sequence $\xi = \{x_0, x_1(b^0), x_2(b^1), \dots, x_N(b^{N-1})\}$ and a basic strategy of player 2 is a sequence $\eta = \{y_0(a^0), y_1(a^1), y_2(a^2), \dots, y_N(a^N)\}$.

3. In chess, possible actions/moves of players 1 and 2 (White and Black) can be identified with positions on the board. When selecting a move, the player selects a new position. The payoffs, depending on the history of play, are defined as follows. If White wins then White gets 1 and Black 0; if Black wins then Black gets 1 and

White 0; in case of a draw, both get 1/2. Illegitimate moves (or sequences of moves) lead, by definition, to a zero payoff for the corresponding player. Winning strategies defined above in the general context correspond to winning strategies in chess, and unbeatable ones to those strategies in chess which guarantee at least a draw.

It is assumed that chess is a *finite-stage* game, which is justified by the following argument. There is a finite number of chess-pieces and a finite number of squares on the board, hence there is a finite number of possible positions. The game automatically terminates as a draw if the same position occurs at least three times, with the same player having to go. Therefore the game cannot last more than N stages, where N is a sufficiently large number.

4. For any history of play h , define $f(h) = u(h) - v(h)$. Before proving Theorem 2, we state two auxiliary propositions.

Proposition 1. *A strategy ξ of player 1 is winning (resp. unbeatable) if for any sequence $b^N = (b_0, \dots, b_N)$ of moves of player 2, $f(h(\xi, b^N)) > 0$ (resp. $f(h(\xi, b^N)) \geq 0$). A strategy η of player 2 is winning (resp. unbeatable) if for any sequence $a^N = (a_0, \dots, a_N)$ of moves of player 1, $f(h(a^N, \eta)) < 0$ (resp. $f(h(a^N, \eta)) \leq 0$).*

Proof. The first assertion follows from the fact that for every strategy profile (ξ, η) the outcome $h(\xi, \eta)$ of the game coincides with $h(\xi, b^N)$ where $b^N = (b_0, \dots, b_N)$ is the sequence of moves of player 2 generated by the strategy profile (ξ, η) . The second assertion is a consequence of the fact that for every strategy profile (ξ, η) the outcome $h(\xi, \eta)$ of the game coincides with $h(a^N, \eta)$ where $a^N = (a_0, \dots, a_N)$ is the sequence of moves of player 1 generated by the strategy profile (ξ, η) . \square

Proposition 2. *One of the assertions (i) - (iii) listed in Theorem 2 holds if and only if the following two conditions are satisfied:*

- (I) *Either player 1 has a winning strategy, or player 2 has an unbeatable strategy.*
- (II) *Either player 2 has a winning strategy, or player 1 has an unbeatable strategy.*

Proof. "If". Suppose (I) and (II) hold. Let us show that one of the assertions (i)-(iii) holds. If (i) is valid, the assertion is proved. Suppose (i) is not valid. Then by virtue of (I), player 2 has an unbeatable strategy. If (ii) holds, the assertion is proved. Suppose not only (i) but also (ii) fails to hold. Then by virtue of (I) and (II), (iii) holds.

"Only if". If (i) (resp. (ii)) holds, then player 1 (resp. player 2) has a winning, and consequently, unbeatable strategy, which implies (I) and (II). Similarly, (iii) yields (I) and (II). \square

5. *Proof of Theorem 2.* According to the duality principle of first-order logic, the propositions

$$\begin{aligned} (\mathbf{P}) \quad & \exists x_0 \forall b_0 \exists x_1(b^0) \forall b_1 \exists x_2(b^1) \dots \forall b_{N-1} \exists x_N(b^{N-1}) \forall b_N : \\ & f(x_0, x_1(b^0), x_2(b^1), \dots, x_N(b^{N-1}), b_0, b_1, b_2, \dots, b_N) > 0 \end{aligned}$$

and

$$\begin{aligned} (\bar{\mathbf{P}}) \quad & \forall a_0 \exists y_0(a^0) \forall a_1 \exists y_1(a^1) \forall a_2 \exists y_2(a^2) \dots \forall a_N \exists y_N(a^N) : \\ & f(a_0, a_1, a_2, \dots, a_N, y_0(a^0), y_1(a^1), y_2(a^2), \dots, y_N(a^N)) \leq 0 \end{aligned}$$

are the *negations* of each other, and so either (\mathbf{P}) or $(\bar{\mathbf{P}})$ holds. If (\mathbf{P}) holds, then player 1 has a winning basic strategy $\{x_0, x_1(b^0), x_2(b^1), \dots, x_N(b^{N-1})\}$. If $(\bar{\mathbf{P}})$ is valid, then player 2 has an unbeatable basic strategy $\{y_0(a^0), y_1(a^1), y_2(a^2), \dots, y_N(a^N)\}$. This proves (I).

To verify (II) consider the function $\phi(r)$ of a real number r defined as -1 if $r < 0$ and as 1 if $r \geq 0$. Clearly $\phi(r) \leq 0$ if and only if $r < 0$ and $\phi(r) > 0$ if and only if $r \geq 0$. Replace $f(h)$ in (\mathbf{P}) and $(\bar{\mathbf{P}})$ by $g(h) = \phi(f(h))$. Repeating the above argument with $g(h)$ in place of $f(h)$, we obtain that player 2 has a winning strategy $\tau = \{y_0(a^0), y_1(a^1), y_2(a^2), \dots, y_N(a^N)\}$ if $(\bar{\mathbf{P}})$ holds and player 1 has an unbeatable strategy $\sigma = \{x_0, x_1(b^0), x_2(b^1), \dots, x_N(b^{N-1})\}$ if (\mathbf{P}) holds. This proves (II). \square

Remark. A classical method for constructing solutions in dynamic games is based on the principle of backward induction – see e.g. [2]. Rather than using this method, we employ, following Kechris [5, pp. 147-148], Khomskii [6, pp. 8-9] and others, elementary techniques of first-order logic, which makes it possible to achieve the goal in a much shorter way.

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