To appear in: Handbook of Post Crisis Financial Modelling (E. Haven et al., eds.), Palgrave MacMillan, December 2015.

Evolutionary Behavioural Finance

Igor V. Evstigneev

University of Manchester - Economics, School of Social Sciences

Thorsten Hens

University of Zurich - Department of Banking and Finance; Norwegian School of Economics and Business Administration (NHH); Swiss Finance Institute (Zurich Center)

Klaus Reiner Schenk-Hoppé

University of Manchester - Economics, School of Social Sciences; Norwegian School of Economics (NHH) - Department of Finance

June 4, 2015

Swiss Finance Institute Research Paper No. 15-16

EVOLUTIONARY BEHAVIOURAL FINANCE[1]

Igor Evstigneev
University of Manchester

Thorsten Hens
Swiss Finance Institute, University of Zurich and Norwegian School of Economics

Klaus Reiner Schenk-Hoppé
University of Manchester and Norwegian School of Economics

Abstract. The paper reviews a new research field that develops evolutionary and behavioural approaches for the modeling of financial markets. The main objective is to create a plausible alternative to the conventional Walrasian equilibrium theory based on the hypothesis of full rationality of market players. Rather than maximizing typically unobservable individual utility functions, traders/investors are permitted to have a whole variety of patterns of strategic behaviour depending on their individual psychology. The models considered in this field combine elements of evolutionary game theory (solution concepts) and stochastic dynamic games (strategic frameworks).

---

[1] This paper grew out of lecture notes prepared for presentations at the UK Mathematical Finance Workshop, King's College, London, June 2013, Trimester Program "Stochastic Dynamics in Economics and Finance" at the Hausdorff Research Institute for Mathematics, University of Bonn, June 2013, ESRC Seminar "Financial Modelling Post 2008", University of Manchester, October 2013, Technion - Israel Institute of Technology, Haifa, November 2013, and London School of Economics, December 2013. The authors are grateful to the organizers of the conferences and seminars for the invitations and to the participants for helpful discussions. Financial support from the Swiss National Science Foundation grant 149856 "Behavioural Financial Markets" (2014-2016) is gratefully acknowledged.

1 The main goals and methodology

Economics and Finance: Global challenges. The creation and protection of financial wealth is one of the most important roles of modern societies. People will confine to work hard and save for future generations only if they can be sure that the efforts they exert every day will be rewarded by a better standard of living. This, however, can only be achieved with a well-functioning financial market. Unfortunately, a breakdown of the financial system as in the great financial crisis of 2007 and 2008 destroys the trust in this important social arrangement. To avoid such crises we need to improve our understanding of financial markets that, so far, has been built on totally unrealistic assumptions about the behaviour of people acting in them. The most fundamental and at the same time the most questionable in modern Economic Theory is the hypothesis of *full rationality* of economic agents who are assumed to maximize their utility functions subject to their individual constraints, or in mathematical language, solve well-defined and precisely stated *constrained optimization problems*.

Evolutionary Behavioural Finance. The general objective of this direction of research is the development of a new interdisciplinary field *Evolutionary Behavioural Finance (EBF)*, that combines behavioural and evolutionary approaches to the modelling of financial markets. The focus of study is on fundamental questions and problems pertaining to Finance and Financial Economics, especially those related to equilibrium asset pricing and portfolio selection. Models of market dynamics and equilibrium that are developed in the framework of EBF provide a plausible alternative to the conventional approach to asset pricing based on the hypothesis of full rationality and are aimed at practical quantitative applications.

The question of price formation in asset markets is central to Financial Economics. Among the variety of approaches addressing this question, one can observe two general and well-established theories: one deals with basic assets and the other focuses on derivative securities. Models for the pricing of derivative securities were developed in the last three or four decades, following the "Black-Scholes revolution". They were based on new ideas, led to the creation of a profound mathematical theory and became indispensable in practice. At the same time, the only general theory explaining the formation of the prices of basic assets, whether stock or equity, appears to be the Arrow-Debreu *General Equilibrium (GE)* analysis in a financial context (Radner [53]). It relies upon the Walrasian paradigm of fully rational utility maximization, going back to Leon Walras, one of the key figures in the economic thought of the 19th century. Although equilibrium models of this kind currently serve as the main framework for teaching and research on asset pricing, they do not provide tools for practical quantitative recommendations and moreover they do not reflect a number of fundamental aspects of modern financial markets. Their crucial drawback is that they do not take into account the enormous variety of patterns of real market behaviour irreducible to individual utility maximization, especially those of an evolutionary nature: growth, domination and survival.

GE theory for the 21st century. EBF develops an alternative equilibrium paradigm, which can be called *Behavioural Equilibrium*, that abandons the hypothesis of full rationality and admits that market participants may have a whole range of patterns of behaviour depending on their individual psychology. Investors' strategies may involve, for example, mimicking, satisficing, and rules of thumb based on experience. They might be *interactive:* depending on the behaviour of the others, and *relative*: taking into account the comparative performance of the others. Objectives of the market participants might be of an *evolutionary* nature: *survival* (especially in crisis environments), *domination* in a market segment, or fastest capital *growth*. The evolutionary aspect is in the main focus of the models developed in the EBF. A synthesis of behavioural and evolutionary approaches makes it possible to obtain rigorous mathematical results identifying strategies that guarantee survival or domination in a competitive market environment.

In the EBF models, the notion of a short-run price equilibrium is defined directly in terms of a strategy profile of the agents, and the process of market dynamics is viewed as a sequence of consecutively related short-run equilibria. Uncertainty on asset payoffs at each period is modelled via an exogenous discrete-time stochastic process governing the evolution of the states of the world. The states of the world are meant to capture various macroeconomic and business cycle variables that may affect investors' behaviour. The traders use general, adaptive strategies (portfolio rules), distributing their current wealth between assets at every period, depending on the observed history of the game and the exogenous random factors. One of the central goals is to identify investment strategies that guarantee the long-run *survival* of any investor using them, in the sense of keeping a strictly positive, bounded away from zero, share of market wealth over the infinite time horizon, irrespective of the investment strategies employed by the other agents in the market. Remarkably, it turns out to be possible to provide a full characterization of such strategies, give explicit formulas for them and show that they are essentially, within a certain class, asymptotically unique.

This approach eliminates a number of drawbacks of the conventional theory. In particular, it does not require the assumption of *perfect foresight* (see Magill and Quinzii [44], p. 36) to establish an equilibrium and most importantly, the knowledge of unobservable individual agents' utilities and beliefs to compute it. It is free of such "curses" of GE as *indeterminacy* of temporary equilibrium and the necessity of *coordination* of plans of market participants, which contradicts the very idea of equilibrium decentralization. It opens new possibilities for the modelling of modern financial markets, in particular on the global level, where objectives of an evolutionary nature play a major role.

The roots of ideas underlying the EBF models lie in *Evolutionary Economics* (Alchian in the 1950s, Nelson and Winter from the 1970s to 1990s), *Behavioural Economics* (Tversky, Kahneman and Smith[2]), and *Behavioural Finance* (Shiller[3]). First mathematical models for EBF were developed during the last decade by the authors of this paper and their collaborators [4-6, 9, 23-30, 35, 36]. This research has already led to a substantial impact in the financial industry [23].

Levels of behavioural modeling. It is important to distinguish between two different methodological levels of behavioural modelling:

• *Individual level*—analyzing individual's behaviour in situations involving risk.

• *Interactive level*—taking into account the dependence of an individual's actions on the actions of others and their influence on market dynamics and equilibrium.

Modern behavioural economics and finance originated from models analyzing an individual's behaviour in situations involving risk. Inspired by the seminal work of Kahneman and Tversky [38], these ideas were developed in finance by Barberis and Thaler [12], Barberis, Huang and Santos [10], Barberis, Shleifer and Vishny [1], Barberis and Xiong [13] and others. According to the classical approach, a decision maker facing uncertainty maximizes expected utility. This hypothesis leads to a number of paradoxes and inconsistencies with reality (Friedman and Savage [3], Allais [3], Ellsberg [22], and Mehra and Prescott [49]). To resolve the paradoxes, the individual-level behavioural approach suggests to replace expected utilities by more general functionals of random variables, based, in particular, on the Kahneman and Tversky [38] *prospect theory* and involving *distorted probabilities* or *capacities* (non-additive measures) [21]. In quantitative finance, studies along these lines have been pursued by Hens, Levy, Zhou, De Giorgi, Rieger, and others [18, 19, 20, 42, 61].

A synthesis of evolutionary and dynamic games. The analysis of market behaviour from the angle of interaction, especially *strategic interaction*, of economic agents is a deeper and more

---

[2] Kahneman and Smith: the 2002 Nobel Laureates in Economics.

[3] The 2013 Nobel Prize in Economics.

advanced aspect of behavioural modelling. This is the primary emphasis of the research area discussed in this article. To build models of strategic behavior in financial markets we propose new mathematical frameworks *combining elements of stochastic dynamic games and evolutionary game theory*.

The main strategic framework of our behavioural equilibrium models is that of stochastic dynamic games (Shapley [57]). However, the emphasis on questions of survival and extinction of investment strategies in a market selection process links our work to evolutionary game theory (Weibull [60], Hofbauer and Sigmund [37]). The latter was designed initially for the modelling of biological systems and then received fruitful applications in economics. The notion of a survival portfolio rule, which is stable with respect to the market selection process, is akin to the notions of evolutionary stable strategies (ESS) introduced by Maynard Smith and Price [46] and Schaffer [55]. However, the mechanism of market selection in our models is radically distinct from the typical schemes of evolutionary game theory, where repeated random matchings of species or agents in large populations result in their survival or extinction in the long run. Standard frameworks considered in that field deal with models based on a given static game, in terms of which the process of evolutionary dynamics is defined. Players in such models follow relatively simple predefined algorithms, which completely describe their behaviour. Our model is quite different in its essence. Although the game solution concept we deal with—a *survival strategy*—is of an evolutionary nature, the notion of a strategy we use is the one which is characteristic for the conventional setting of dynamic stochastic games. A strategy in this setting is a general rule prescribing what action to take based on the observation of all the previous play and the history of random states of the world. Players are allowed to use any rule of this kind, possess all information needed for this purpose and have a clear goal: guaranteed survival. Thus, the model at hand connects two basic paradigms of game theory: evolutionary and dynamic games [5, 6].

Unbeatable strategies. The present game-theoretic framework has the following remarkable feature. One can equivalently reformulate the solution concept of a survival strategy in terms of the wealth process of a player, rather than in terms of his market share process. A strategy guarantees survival if and only if it guarantees the fastest asymptotic growth of wealth (almost surely) of the investor using it. This can be expressed by saying that the strategy is *unbeatable* in terms of the growth rate of wealth.

Nowadays, Nash equilibrium is the most common game solution concept. However, in the early days of game theory, the idea of an *unbeatable* (or *winning*) strategy was central to the field. At those times, solving a game meant primarily finding a winning strategy. This question was considered in the paper by Bouton [16], apparently the earliest mathematical paper in the field. Borel [15] wrote: "One may propose to investigate whether it is possible to determine a method of play better than all others; i.e. one that gives the player who adopts it a superiority over every player who does not adopt it." It is commonly viewed that finding an unbeatable strategy is a problem of extreme complexity that can be solved only in some exceptional cases for some artificially designed games such as the Bouton's game "Nim". However, in our practice-motivated context, this problem does have a solution, and this is apparently one of the first, if not the first, result of this kind possessing quantitative real-world applications.

The remainder of this paper is organized as follows. In Sections 2 and 3 we present the basic EBF model and the main results related to it. Section 4 focuses on a simplified version of the basic model. In the last section we discuss some open problems and directions of further research.

2 The basic model

The data of the model. In this section we present (in a somewhat simplified form) the main EBF model and key results related to it. Let $s_t \in S$ $(t = 1, 2, ...)$ be a stochastic process with values in a measurable space S. Elements in S are interpreted as *states of the world*, s_t being the state at date t. In the market under consideration, K *assets* $k = 1, ..., K$, are traded. At date t one unit of asset k pays *dividend* $D_{t,k}(s^t) \geq 0$ depending on the history $s^t = (s_1, ..., s_t)$ of states of the world by date t. It is assumed that

$$\sum_{k=1}^{K} D_{t,k}(s^t) > 0 \text{ for all } t, s^t,$$

$$ED_{t,k}(s^t) > 0,\ k = 1, ..., K,$$

where E is the expectation with respect to the underlying probability P. Thus, at least one asset pays a strictly positive dividend at each state of the world and the expected dividends for all the assets are strictly positive.

Asset supply is exogenous: the total mass (the number of "physical units") of asset k available at date t is $V_{t,k} = V_{t,k}(s^t)$.

Investors and their portfolios. There are N *investors (traders)* $i \in \{1, ..., N\}$. Every investor i at each time $t = 0, 1, 2, ...$ selects a *portfolio*

$$x_t^i = (x_{t,1}^i, ..., x_{t,K}^i) \in \mathbb{R}_+^K,$$

where $x_{t,k}^i$ is the number of units of asset k in the portfolio x_t^i. The portfolio x_t^i for $t \geq 1$ depends, generally, on the current and previous states of the world:

$$x_t^i = x_t^i(s^t),\ s^t = (s_1, ..., s_t).$$

Asset prices. We denote by $p_t \in \mathbb{R}_+^K$ the vector of market prices of the assets. For each $k = 1, ..., K$, the coordinate $p_{t,k}$ of $p_t = (p_{t,1}, ..., p_{t,K})$ stands for the price of one unit of asset k at date t. The scalar product

$$\langle p_t, x_t^i \rangle := \sum_{k=1}^{K} p_{t,k} x_{t,k}^i$$

expresses in terms of the prices $p_{t,k}$ the value of the investor i's portfolio x_t^i at date t.

The state of the market at each date t is characterized by a set of vectors

$$(p_t, x_t^1, ..., x_t^N),$$

where p_t is the vector of asset prices and $x_t^1, ..., x_t^N$ are the portfolios of the investors.

Investors' budgets. At date $t = 0$ investors have initial endowments: amounts of cash $w_0^i > 0$ ($i = 1, 2, ..., N$). These initial endowments form the traders' budgets at date 0. Trader i's budget at date $t \geq 1$ is

$$B_t^i(p_t, x_{t-1}^i) := \langle D_t(s^t) + p_t, x_{t-1}^i \rangle,$$

where

$$D_t(s^t) := (D_{t,1}(s^t), ..., D_{t,K}(s^t)).$$

It consists of two components: the dividends $\langle D_t(s^t), x_{t-1}^i \rangle$ paid by the yesterday's portfolio x_{t-1}^i and the market value $\langle p_t, x_{t-1}^i \rangle$ of the portfolio x_{t-1}^i expressed in terms of the today's prices p_t.

Investment rate. A fraction $\sim$ of the budget is invested into assets. We will assume that the *investment rate* $\sim \in (0,1)$ is a fixed number, the same for all the traders. The number $1 - \sim$ can represent, e.g., the *tax rate* or the *consumption rate*. The assumption that $1 - \sim$ is the same for all the investors is quite natural in the former case. In the latter case, it might seem restrictive, but in the present context it is indispensable since we focus in this work on the analysis of the comparative performance of trading strategies (portfolio rules) in the long run. Without this assumption, an analysis of this kind does not make sense: a seemingly worse performance of a portfolio rule might be simply due to a higher consumption rate of the investor.

Investment proportions. For each $t \geq 0$, each trader $i = 1, 2, ..., N$ selects a vector of *investment proportions* $\mathrm{a}_t^i = (\mathrm{a}_{t,1}^i, ..., \mathrm{a}_{t,K}^i)$ according to which he/she plans to distribute the available budget between assets. Vectors a_t^i belong to the unit simplex

$$\Delta^K := \{(a_1, ..., a_K) \geq 0 \ : \ a_1 + ... + a_K = 1\}.$$

The vectors a_t^i represent the players' (investors') *actions* or *decisions*.

Investment strategies (portfolio rules). How do investors select their investment proportions? To describe this we use a game-theoretic approach: decisions of players are specified by their strategies. The notion of a (pure) strategy we use is standard for stochastic dynamic games. *A strategy in a stochastic dynamic game is a rule prescribing how to act based on information about all the previous actions of the player and his rivals, as well as information about the observed random states of the world.*

A formal definition is as follows. A *strategy (portfolio rule)* Λ^i of investor i is a sequence of measurable mappings

$$\Lambda_t^i(s^t, H^{t-1}), \ t = 0, 1, ...,$$

assigning to each history $s^t = (s_1, ..., s_t)$ of states of the world and each *history of the game*

$$H^{t-1} := \{\mathrm{a}_m^i \ : \ i = 1, ..., N, \ m = 0, ..., t-1\}$$

the vector of investment proportions $\mathrm{a}_t^i = \Lambda_t^i(s^t, H^{t-1})$.

Since the sets of investors' portfolios

$$x_0, x_1, ..., x_{t-1}, \ x_l = (x_l^1, ..., x_l^N),$$

and the equilibrium prices $p_0, ..., p_{t-1}$ are determined by the vectors of investment proportions a_m^i, $i = 1, ..., N$, $m = 0, ..., t-1$, the history of the game contains information about the whole *market history* $(p_0, x_0), \ldots, (p_{t-1}, x_{t-1})$.

Basic strategies. Among general portfolio rules, we will distinguish those for which Λ_t^i

depends only on s^t and does not depend on the market history $(p^{t-1}, x^{t-1}, \mathrm{a}^{t-1})$. Clearly, they require substantially less information than general strategies! We will call such portfolio rules *basic*. They play an important role in the present work: the survival strategy we construct belongs to this class.

Investor i**'s demand function.** Given a vector of investment proportions $\mathrm{a}_t^i = (\mathrm{a}_{t,1}^i, \ldots, \mathrm{a}_{t,K}^i)$ of investor i, his demand function is

$$X_{t,k}^i(p_t, x_{t-1}^i) = \frac{\sim \mathrm{a}_{t,k}^i B_t^i(p_t, x_{t-1}^i)}{p_{t,k}}.$$

where $\sim$ is the investment rate.

Equilibrium and dynamics. We examine the *equilibrium market dynamics*, assuming that, in each time period, aggregate demand for each asset is equal to its supply:

$$\sum_{i=1}^N X_{t,k}^i(p_t, x_{t-1}^i) = V_{t,k}, \ k = 1, \ldots, K.$$

(Recall that asset supply is exogenous and equal to $V_{t,k}$.)

Asset market dynamics can be described in terms of portfolios and prices by the equations:

$$p_{t,k} V_{t,k} = \sum_{i=1}^N \sim \mathrm{a}_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle, \ k = 1, \ldots, K; \tag{1}$$

$$x_{t,k}^i = \frac{\sim \mathrm{a}_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle}{p_{t,k}}, \ k = 1, \ldots, K, \ i = 1, 2, \ldots, N. \tag{2}$$

(All the variables with subscript t depend on s^t.) The vectors $\mathrm{a}_t^i = (\mathrm{a}_{t,k}^i)$ are determined recursively by the given strategy profile $(\Lambda^1, \ldots, \Lambda^N)$:

$$\mathrm{a}_t^i(s^t) := \Lambda_t^i(s^t, p^{t-1}, x^{t-1}, \ \mathrm{a}^{t-1}).$$

The pricing equation (1) has a unique solution $p_{t,k} \geq 0$ if $V_{t,k} \geq V_{t-1,k}$ (growth), or under a weaker assumption: $\sim V_{t-1,k} / V_{t,k} < 1$. At date $t = 0$, the budgets involved in the above formulas are the given initial endowments $w_0^i > 0$.

Admissible strategy profiles. We will consider only *admissible* strategy profiles: those for which aggregate demand for each asset is always strictly positive. This guarantees that $p_{t,k} > 0$ (only in this case the above formula for $x_{t,k}^{i}$ makes sense). The focus on such strategy profiles will not lead to a loss in generality in the context of this work: at least one of the strategies we deal with always has strictly positive investment proportions, which guarantees admissibility.

Market shares of the investors. We are mainly interested in comparing the long-run performance of investment strategies described in terms of market shares of the investors. Investor i's *wealth* at time t is

$$w_t^i = \langle D_t(s^t) + p_t, x_{t-1}^i \rangle$$

(dividends + portfolio value). Investor i's *relative wealth,* or i's *market share,* is

$$r_t^i = \frac{w_t^i}{w_t^1 + \ldots + w_t^N}.$$

The dynamics of the vectors $r_t = (r_t^1, \ldots, r_t^N)$ are described by the random dynamical system

$$r_{t+1}^i = \sum_{k=1}^{K} [\lambda \langle \alpha_{t+1,k}, r_{t+1} \rangle + (1-\lambda) R_{t+1,k}] \frac{\alpha_{t,k}^i r_t^i}{\langle \alpha_{t,k}, r_t \rangle}, \tag{3}$$

$i = 1, \ldots, N$, $t \geq 0$, where

$$R_{t,k} = R_{t,k}(s^t) := \frac{D_{t,k} V_{t-1,k}}{\sum_{m=1}^{K} D_{t,m} V_{t-1,m}} \tag{4}$$

(relative dividends). Equations (3), following from (1) and (2), make it possible to determine $r_{t+1} = (r_{t+1}^1, \ldots, r_{t+1}^N)$ based on $r_t = (r_t^1, \ldots, r_t^N)$ and thus generate a random sequence $r_0, r_1, r_2, \ldots$ of the vectors of market shares of the investors.

Survival strategies. Given an admissible strategy profile $(\Lambda^1, \ldots, \Lambda^N)$, we say that the portfolio rule Λ^1 (or the investor 1 using it) *survives* with probability one if

$$\inf_{t\geq 0} r_t^1 > 0$$

almost surely (a.s.). This means that for almost all realizations of the process of states of the world (s_t), the market share of the first investor is bounded away from zero by a strictly positive random variable.

A portfolio rule Λ^1 is called a *survival strategy* if investor 1 using it survives with probability one *irrespective of what portfolio rules are used by the other investors* (as long as the strategy profile is admissible).

A central goal is to identify survival strategies. The main results obtained in this direction are outlined in the next section.

Marshallian temporary equilibrium. Some comments regarding the model are in order. The present model revives in a new context the *Marshallian* concept of temporary equilibrium. Our description of the dynamics of the asset market follows the ideas outlined (in the context of commodity markets) in the classical treatise by Alfred Marshall [45] ”Principles of Economics”, Book V, Chapter II “Temporary Equilibrium of Demand and Supply”. This notion of temporary equilibrium is different from the one going back to Hicks and Lindahl (1930-40s), which has prevailed in the GE literature in the 1970-90s (e.g. Grandmont and Hildenbrand [34] and Grandmont [33]). The former may be regarded as "equilibrium in actions", while the latter as "equilibrium in beliefs”; for a comparative discussion of these approaches see Schlicht [56].

In the model we deal with, the dynamics of the asset market is modeled in terms of a sequence of temporary equilibria. At each date t the investors' strategies $\alpha^i_{t,k}$, the asset dividends $D_k(s^t)$ and the portfolios x^i_{t-1} determine the asset prices $p_t = (p^1_t,..., p^K_t)$ equilibrating asset demand and supply. The asset holdings $x^i_{t-1} = (x^i_{t-1,1},..., x^i_{t-1,K})$ play the role of initial endowments available at the beginning of date t. The portfolios x^i_t selected by the agents in accordance with their demand functions are transferred to date $t+1$ and then in turn serve as initial endowments for the investors.

The dynamics of the asset market described above are similar to the dynamics of the commodity market as outlined in the classical treatise by Alfred Marshall [45]. Marshall's ideas were introduced into formal economics by Samuelson [54].

Samuelson's hierarchy of equilibrium processes. As it was noticed by Samuelson [54], in order to study the process of market dynamics by using the Marshallian "moving equilibrium method," one needs to distinguish between at least two sets of economic variables changing with different speeds. Then the set of variables changing slower (in our case, the set of vectors of the traders' investment proportions) can be temporarily fixed, while the other (in our case, the asset prices p_t) can be assumed to rapidly reach the unique state of partial equilibrium. Samuelson [54] writes about this approach:

> I, myself, find it convenient to visualize equilibrium processes of quite different speed, some very slow compared to others. Within each long run there is a shorter run, and within each shorter run there is a still shorter run, and so forth in an infinite regression. For analytic purposes it is often convenient to treat slow processes as data and concentrate upon the processes of interest. For example, in a short run study of the level of investment, income, and employment, it is often convenient to assume that the stock of capital is perfectly or sensibly fixed.

As it follows from the above citation, Samuelson thinks about a hierarchy of various equilibrium processes with different speeds. In our model, it is sufficient to deal with only two levels of such a hierarchy.

Continuous vs discrete time. The above approach to the modeling of equilibrium and dynamics of financial markets requires discretization of the time parameter. The time interval under consideration has to be divided into subintervals during which the "slow" variables must be kept frozen, while the "fast" ones rapidly reach a unique state of equilibrium. In this connection, discrete-time settings in our field are most natural for the modelling purposes, and attempts to realize similar ideas in continuous-time frameworks face serious conceptual and technical difficulties [51, 52].

3 The main results

Assumption 1. Assume that the total mass of each asset grows (or decreases) at the same constant rate $¥ > \sim$:

$$V_{t,k} = ¥^t V_k \ ,$$

where V_k ($k = 1, 2, ..., K$) are the initial amounts of the assets. In the case of real assets—involving long-term investments with dividends (e.g., real estate, transport, communications, media, etc.) —the above assumption means that the economy under consideration is on a *balanced growth path*.

Relative dividends. Under the above assumption, the *relative dividends* of the assets $k = 1, ..., K$ (see (4)) can be written as

$$R_{t,k} = R_{t,k}(s^t) := \frac{D_{t,k}(s^t)V_k}{\sum_{m=1}^{K} D_{t,m}(s^t)V_m}, \ k = 1, ..., K, \ t \geq 1.$$

We denote by $R_t(s^t) = (R_{t,1}(s^t), ..., R_{t,K}(s^t))$ the vector of the relative dividends of the assets $k = 1, 2, ..., K$.

Definition the survival strategy Λ^*. Put

$$° := \sim / ¥ , \ °_t := °^{t-1}(1 - °)$$

and consider the portfolio rule Λ^* with the vectors of investment proportions

$$a_t^*(s^t) = (a_{t,1}^*(s^t), ..., a_{t,K}^*(s^t)),$$

$$a_{t,k}^* = E_t \sum_{l=1}^{\infty} °_l R_{t+l,k} \ , \tag{5}$$

where $E_t(\cdot) = E(\cdot \mid s^t)$ is the conditional expectation given s^t ; $E_0(\cdot)$ is the unconditional expectation $E(\cdot)$.

Assumption 2. There exists a constant $\text{¢} > 0$ such that for all k and t we have

$$E_t R_{t+1,k}(s^{t+1}) > \text{¢} \text{ (a.s.).}$$

This assumption implies that the conditional expectation in the definition of $\text{a}^*_{t,k}$, which is not less than $(1-{}^\circ)E(R_{t+1,k} \mid s^t)$, is strictly positive a.s., and so we can select a version of this conditional expectation that is strictly positive for all s^t. This version will be used in the definition of the strategy Λ^*. It follows from the strict positivity of $\text{a}^*_{t,k}$ that any strategy profile containing Λ^* is admissible.

A central result is as follows [5, Theorem 1]:

Theorem 1. *The portfolio rule* Λ^* *is a survival strategy.*

The meaning of Λ^*. The portfolio rule Λ^* defined by (5) combines three general principles in Financial Economics.

(a) Λ^* prescribes the allocation of wealth among assets in the proportions of their *fundamental values*: the expectations of the flows of the discounted future dividends.

(b) The strategy Λ^*, defined in terms of the *relative (weighted) dividends***,** is analogous to the CAPM strategy involving investment in the *market portfolio.*[4]

(c) The portfolio rule Λ^* is related (and in some special cases reduces, see Section 4) to the *Kelly portfolio rule* prescribing to maximize the expected logarithm of the portfolio return.

Note that the main strength of the result obtained lies in the fact that the basic strategy Λ^*, requiring information only about the exogenous process of states of the world, survives in competition against

[4]However, it should be emphasized that instead of weighing assets according to their prices, in Λ^* the weights are based on fundamentals. In practice, Λ^* is an example of *fundamental indexing* (Arnott, Hsu and West [8]).

any, not necessarily basic, strategies of the rivals, that might use all possible information about the market history and the previous actions of all the players.

Asymptotic uniqueness. As we noted, the portfolio rule Λ^* belongs to the class of basic portfolio rules: the investment proportions $a_t^*(s^t)$ depend only on the history s^t of the process of states of the world, and do not depend on the market history. The following theorem [5, Theorem 2] shows that in this class the survival strategy $\Lambda^* = (a_t^*)$ is essentially unique: any other basic survival strategy is asymptotically similar to Λ^*.

Theorem 2. *If* $\Lambda = (a_t)$ *is a basic survival strategy, then*

$$\sum_{t=0}^{\infty} \| a_t^* - a_t \|^2 < \infty \text{ (a.s.).}$$

Here, we denote by $\|\cdot\|$ the Euclidean norm in a finite-dimensional space. Theorem 2 is akin to various *turnpike* results in the theory of economic dynamics, expressing the idea that all optimal or asymptotically optimal paths of an economic system follow in the long run essentially the same route: the turnpike (Nikaido [50], McKenzie [48]). Theorem 2 is a direct analogue of Gale's turnpike theorem for "good paths" (Gale [32]); for a stochastic version of this result see Arkin and Evstigneev [7]).

The i.i.d. case. If $s_t \in S$ are *independent and identically distributed (i.i.d.)*, then the investment proportions

$$a_{t,k}^* = a_k^* = ER_k(s_t),$$

do not depend on t, and so Λ^* is a *fixed-mix (constant proportions) strategy*. Furthermore, a^* is independent of the investment rate $\sim$. The most important feature of this result is that it indicates a constant proportions strategy which survives in competition against any strategies with variable investment proportions depending on all information about the history of the game.

Global evolutionary stability of Λ^*. Consider the i.i.d. case in more detail. This case is important for quantitative applications and admits a deeper analysis of the model. Let us concentrate on fixed-mix (constant proportions) strategies. In the class of such strategies, Λ^* is *globally evolutionarily stable* [26, Theorem 1]:

Theorem 3. *If among the* N *investors, there is a group using* Λ^*, *then those who use* Λ^* *survive, while all the others are driven out of the market (their market shares tend to zero a.s.).*

In order to survive you have to win! One might think that the focus on survival substantially restricts the scope of the analysis: "one should care of survival only if things go wrong". It turns out, however, that the class of survival strategies coincides with the class of *unbeatable* strategies performing *in terms of wealth accumulation* in the long run not worse than any other strategies competing in the market. *Thus in order to survive you have to win!*

To be more precise let us call a strategy Λ *unbeatable* if it has the following property. Suppose investor i uses the strategy Λ, while all the others $j \neq i$ use *any* strategies. Then the wealth process w_t^j of every investor $j \neq i$ cannot grow asymptotically faster than the wealth process w_t^i of investor i: $w_t^j \leq H w_t^i$ (a.s.) for some random constant H.

It is an easy exercise to show that *a strategy is a survival strategy if and only if it is unbeatable.*

Unbeatable strategies: a general definition [6]**.** Consider an abstract game of N players $i = 1,..., N$ selecting strategies Λ^i in some sets $\mathcal{L}^i$. Let $w^i = w^i(\Lambda^1,..., \Lambda^N) \in \mathcal{W}$ be the *outcome* of the game for player i given the strategy profile $(\Lambda^1,..., \Lambda^N)$. Suppose a preference relation

$$(w^j) \leq (w^i) \quad (w^i, w^j \in \mathcal{W})$$

is given, comparing relative performance of players i and j. A strategy Λ of player i is termed *unbeatable* if for any admissible strategy profile $(\Lambda^1, \Lambda^2,..., \Lambda^N)$ in which $\Lambda^i = \Lambda$, we have

$$w^j(\Lambda^1, \Lambda^2, ..., \Lambda^N) \le w^i(\Lambda^1, \Lambda^2, ..., \Lambda^N) \text{ for all } j \ne i.$$

Thus, if player i uses Λ, he *cannot be outperformed by any of the rivals* $j \ne i$, *irrespective of what strategies they employ.*

Unbeatable strategies of capital accumulation. In our model, an outcome of the game for player i is the random wealth process $w^i = (w_t^i)$. The preference relation $\preceq$ is introduced as follows. For two sequences of positive random numbers (w_t^i) and (w_t^j), we define

$$(w_t^j) \preceq (w_t^i) \text{ iff } w_t^j \le H w_t^i \text{ (a.s.)}$$

for some random $H > 0$. The relation $(w_t^j) \preceq (w_t^i)$ means that (w_t^j) does not grow asymptotically faster than (w_t^i) (a.s.).

Unbeatable strategies and evolutionary game theory. The basic solution concepts in evolutionary game theory – *evolutionary stable strategies* (Maynard Smith and Price [46], Maynard Smith [47], Schaffer [55]) – may be regarded as *conditionally* unbeatable strategies (the number of mutants is small enough, or they are identical). Unconditional versions of the standard ESS were considered by Kojima [40].

4 A version of the basic model: Short-lived assets

Short-lived assets. We present a simplified version of the basic model in which assets "live" only one period. This model is more amenable for mathematical analysis and makes it possible to develop a more complete and transparent theory. It has often served as a "proving ground" for testing new conjectures regarding the basic one. Finally, it clearly demonstrates links of the present line of studies to some adjacent fields of research such as the classical capital growth theory with exogenous asset prices (Kelly [39], Latané [41], Thorp [59], Algoet and Cover [2], MacLean et al. [43]).

There are K assets/securities $k = 1, 2, ..., K$. They are issued at the beginning of each time interval $t-1, t$, yield payoffs $A_{t,k}(s^t)$ at the end of it and then expire. They are identically "re-born" at the next date t, and the cycle repeats. Asset supply at date t is $V_{t,k}(s^t) > 0$. It is assumed that

$$\sum_{k=1}^{K} A_{t,k}(s^t) > 0 \text{ for all } t \text{ and } s.$$

Investors, portfolios and prices. *Investors/players* $i = 1,..., N$ construct *portfolios* $x_t^i = (x_{t,1}^i,..., x_{t,K}^i)$ by selecting vectors of *investment proportions* $\mathrm{a}_t^i = (\mathrm{a}_{t,1}^i,..., \mathrm{a}_{t,K}^i) \in \Delta^K$. The number $\mathrm{a}_{t,1}^i$ indicates the fraction of the *budget*

$$\langle A_t, x_{t-1}^i \rangle,\ A_t(s^t) := (A_{t,1}(s^t),..., A_{t,K}(s^t)),$$

of investor i allocated to asset k. Note that in contrast with the basic model, *all* the budget is used for investment. The budget at date $t = 0$ is the *initial endowment* $w_0^i > 0$.

Investors' portfolios $x_t^i = (x_{t,1}^i,..., x_{t,K}^i)$ are expressed as

$$x_{t,k}^i = \frac{\mathrm{a}_{t,k}^i \langle A_t, x_{t-1}^i \rangle}{p_{t,k}},$$

and *equilibrium asset prices* $p_t = (p_{t,1},..., p_{t,K})$ are obtained from the market clearing condition (supply = demand):

$$\sum_{i=1}^{N} \frac{\mathrm{a}_{t,k}^i \langle A_t, x_{t-1}^i \rangle}{p_{t,k}} = V_k \ , k = 1, 2,..., K.$$

Thus

$$p_{t,k} = \sum_{i=1}^{N} \frac{\mathrm{a}_{t,k}^i \langle A_t, x_{t-1}^i \rangle}{V_k}.$$

Strategies and market dynamics. Vectors of investment proportions a_t^i are selected by investors $i = 1,..., N$ according to *strategies*

$$\Lambda_t^i(s^t, \mathrm{a}^{t-1}), t = 1, 2, \ldots$$

depending on the history of states of the world $s^t = (s_1, \ldots, s_t)$ and the history of the game

$$\mathrm{a}^{t-1} := (\mathrm{a}_l^i),\ i = 1, \ldots, N,\ l = 0, \ldots, t-1.$$

Basic strategies depend only on s^t, and do not depend on a^{t-1}. A strategy profile of investors generates, like in the basic model, wealth processes of the investors

$$w_t^i = w_t^i(s^t) := \langle A_t(s^t), x_{t-1}^i(s^{t-1}) \rangle,\ i = 1, 2, \ldots, N,$$

which in turn determine the dynamics of their market shares

$$r_t^i := w_t^i / w_t\ ,\ w_t := \sum_{i=1}^{N} w_t^i.$$

In the present model, the dynamics of the vectors of investors' market shares $r_t = (r_t^i, \ldots, r_t^N)$ is governed by the random dynamical system

$$r_{t+1}^i = \sum_{k=1}^{K} R_{t+1,k} \frac{\mathrm{a}_{t,k}^i r_t^i}{\langle \mathrm{a}_{t,k}, r_t \rangle},\ i = 1, \ldots, N, , \qquad (6)$$

which is substantially simpler than (3), in particular, because r_{t+1} is obtained from r_t by an explicit formula (in contrast with the implicit relation (3)). It should be noted that (3) reduces to (6) when $\sim = 0$. The intuitive meaning of this fact is clear: in the short-lived asset case one cannot reinvest. Assets live only one period and tomorrow's assets are not the same as today's ones.

Results. Define the *relative payoffs* by

$$R_{t,k}(s^t) := \frac{A_{t,k}(s^t) V_{t-1,k}(s^{t-1})}{\sum_{m=1}^{K} A_{t,m}(s^t) V_{t-1,m}(s^{t-1})},$$

and put $R_t(s^t) = (R_{t,1}(s^t), \ldots, R_{t,K}(s^t))$. Consider the basic strategy $\Lambda^* = (\mathrm{a}_t^*)$ defined by

$$\mathrm{a}_t^*(s^t) := E_t R_{t+1}(s^{t+1}),$$

where $E_t(\cdot) = E(\cdot \mid s^t)$ is the conditional expectation given s^t. Assume

$$E \ln E_t R_{t+1,k}(s^{t+1}) > -\infty.$$

Theorems 1-3, reformulated literally for the present model, are valid (see [6, 2]):

Theorem 4. *The portfolio rule* Λ^* *is a survival strategy. It is asymptotically unique in the class of basic strategies. In the i.i.d. case, it is globally asymptotically stable.*

Betting your beliefs. The strategy Λ^* prescribes to invest in accordance with the proportions of the (conditionally) expected relative payoffs. This investment principle is sometimes referred to as "betting your beliefs". The same principle is in a sense valid in the basic model, see the definition of Λ^* in the previous section.

Horse race model. Consider the following toy model of an asset market (cf. Kelly [39], Blume and Easley [14]). The state space S consists of K elements: $S = \{1, 2, ..., K\}$, $A_k(s) = 0$ if $s \neq k$ and $A_k(s) = 1$ if $s = k$, and $V_{t,1} = V_{t,2} = ... = V_{t,K} = 1$. Thus there are as many states of the world as there are assets, and one and only one asset yields unit payoff in each state of the world. Assets with this payoff structure are called *Arrow securities*.

One can think of this model as describing a sequence of horse races with independent outcomes. Only one horse k wins in each race yielding unit payoff. This event occurs with probability $\text{®}_k = P\{s_t = k\}$. In this example, the relative payoffs $R_k(s)$ coincide with $A_k(s)$, and the strategy $\Lambda^* = (a^*)$ of "betting your beliefs" takes on the form:

$$a^* = (a^*_1, ..., a^*_K), \text{ where } a^*_k = ER_k(s_t) = p_k.$$

The strategy Λ^* and the Kelly portfolio rule. It is well-known and easy to prove that the function

$$\Phi(a) = E \ln \sum_k R_k(s_t) a_k = \sum_k p_k \ln a_k$$

attains its maximum over $a \in \Delta^K$ at $a^* = (p_1, ..., p_K)$. The investment strategy maximizing the expected logarithm of the portfolio return is called the *Kelly portfolio rule* (Kelly [39], Latané [41], Thorp [59], Algoet and Cover [2], MacLean et al. [43]). Thus in the example under consideration, the strategy Λ^* coincides with the Kelly rule. It is important to emphasize that this is a specific feature of the particular case under consideration. In the general case, Λ^* is a solution to a certain game, rather than a single-player optimization problem, and a direct counterpart of the Kelly rule does not exist.

5 Problems and prospects

We list several major topics for further research.

• Developing EBF models with endogenous asset supply, short selling and leverage.

• Constructing "hybrid" models in which assets with endogenous equilibrium prices, as well as assets with exogenous prices, are traded. The role of an asset of the latter type can be played, e.g., by cash with an exogenous (random or non-random) interest rate. Some progress in the analysis of such models was made in [2].

• Developing "overlapping generations" models with a countable number of assets $k = 1, 2, ...$ each of which has its own life cycle starting from some moment of time $\pm_k$ and terminating at some later moment of time $^2{}_k$.

• Introducing the dependence of the dividends paid off at the end of the time period on the equilibrium prices, and consequently, on the total investment in the asset expressed in terms of these prices.

• Obtaining quantitative results on the *rates* of survival and extinction of portfolio rules in the spirit of those in [9].

• Using the dynamic frameworks which are considered in EBF in more traditional settings: in models with finite time horizons and conventional solution concepts (utility maximization, Nash equilibrium).

• Introducing transaction costs and portfolio constraints into EBF models.

• Creating a *universal* version of EBF, that does not assume the knowledge of underlying probability distributions, similar to the theory of Cover's [17] *universal portfolios.*

• Conducting a systematic analysis of the notion of an unbeatable strategy in a modern game-theoretic perspective.

The above problems constitute a vast research programme requiring substantial efforts during a considerable time period. This programme might need for its realization the development of new conceptual ideas, modelling approaches and mathematical techniques. We do not expect in the nearest perspective a significant progress in all of the above directions of research, but we do expect substantial achievements in several of them, where some preliminary results have already been obtained.

References

[1] Alchian, A.A. (1950) Uncertainty, Evolution, and Economic Theory. Journal of Political Economy 58, 211–221.

[2] Algoet, P.H., and T.M. Cover (1988) Asymptotic Optimality and Asymptotic Equipartition Properties of Log-optimum Investment. Annals of Probability 16, 876–898.

[3] Allais, M. (1953) Le comportement de l'homme rationnel devant le risque, critique des postulats et axiomes de l'ecole americaine. Econometrica 21, 503–546.

[4] Amir, R., Evstigneev, I.V., Hens, T., and K.R. Schenk-Hoppé (2005) Market Selection and Survival of Investment Strategies. Journal of Mathematical Economics 41 *(Special Issue on Evolutionary Finance)*, 105–122.

[5] Amir, R., Evstigneev, I.V., Hens, T., and L. Xu (2011) Evolutionary Finance and Dynamic Games. Mathematics and Financial Economics 5 *(Special Issue on Stochastic Financial Economics)*, 161–184.

[6] Amir, R., Evstigneev, I.V., and K.R. Schenk-Hoppé (2013) Asset Market Games of Survival: A Synthesis of Evolutionary and Dynamic Games. Annals of Finance 9, 121–144.

[7] Arkin V.I., and I.V. Evstigneev (1987) Stochastic Models of Control and Economic Dynamics, Academic Press.

[8] Arnott, R.D., Hsu, J.C., and J.M. West (2008) The Fundamental Index: A Better Way to Invest, Wiley.

[9] Bahsoun, W., Evstigneev, I.V., and L. Xu (2011) Almost Sure Nash Equilibrium Strategies in Evolutionary Models of Asset Markets. Mathematical Methods of Operations Research 73, 235–250.

[10] Barberis, N., Huang, M., and T. Santos (2001) Prospect Theory and Asset Prices. Quarterly Journal of Economics 116, 1–53.

[11] Barberis, N., Shleifer, A., and R. Vishny (1998) A Model of Investor Sentiment. Journal of Financial Economics 49, 307–343.

[12] Barberis, N., and R. Thaler (2003) A Survey of Behavioral Finance. In: *Handbook of the Economics of Finance* (Constantinides, G.M., Harris, M., and R. Stulz, eds.), Elsevier.

[13] Barberis, N., and W. Xiong (2009) What Drives the Disposition Effect? An Analysis of a Long-Standing Preference-Based Explanation. Journal of Finance 64, 751–784.

[14] Blume, L., and D. Easley (1992) Evolution and Market Behavior. Journal of Economic Theory 58, 9–40.

[15] Borel, E. (1953) The Theory of Play and Integral Equations with Skew Symmetric Kernels. Econometrica 21, 97–100.

[16] Bouton, C.L. (1901-1902) Nim, A Game with a Complete Mathematical Theory. Annals of Mathematics (Second Series) 3, No. 1/4, 35–39.

[17] Cover, T.M. (1991) Universal portfolios. Mathematical Finance 1, 1–29.

[18] De Giorgi, E., and T. Hens (2006) Making Prospect Theory Fit for Finance. Financial Markets and Portfolio Management 20, 339–360.

[19] De Giorgi, E., Hens, T., and M.O. Rieger (2010) Financial Market Equilibria with Cumulative Prospect Theory. Journal of Mathematical Economics 46 *(Mathematical Economics: Special Issue in honour of Andreu Mas-Colell, Part 1),* 633–651

[20] De Giorgi, E., and S. Legg (2012) Dynamic Portfolio Choice and Asset Pricing with Narrow Framing and Probability Weighting. Journal of Economic Dynamics and Control 36, 951–972.

[21] Denneberg, D. (1994) Non-additive measure and integral, Kluwer Academic Publishers, Dordrecht.

[22] Ellsberg D. (1961) Risk, ambiguity and the Savage axioms. Quarterly Journal of Economics 75, 643–669.

[23] Evstigneev, I.V. (2014) Mathematical Behavioural Finance: Applications in the Financial Industry, *UK Research Excellence Framework 2014*, *Impact Case Study*, *University of Manchester*. DOI: 10.3927/265533

[24] Evstigneev, I.V., Hens, T., and K.R. Schenk-Hoppé (2002) Market Selection of Financial Trading Strategies: Global Stability. Mathematical Finance 12, 329–339.

[25] Evstigneev, I.V., Hens, T., and K.R. Schenk-Hoppé (2006) Evolutionary Stable Stock Markets. Economic Theory 27, 449–468.

[26] Evstigneev, I.V., Hens, T., and K.R. Schenk-Hoppé (2008) Globally Evolutionarily Stable Portfolio Rules. Journal of Economic Theory 140, 197–228.

[27] Evstigneev, I.V., Hens, T., and K.R. Schenk-Hoppé (2009) Evolutionary Finance. In: *Handbook of Financial Markets: Dynamics and Evolution* (Hens, T., and K.R. Schenk-Hoppé, eds.), pp. 507–566, North-Holland.

[28] Evstigneev, I.V., Hens, T., and K.R. Schenk-Hoppé (2011) Local Stability Analysis of a Stochastic Evolutionary Financial Market Model with a Risk-free Asset. Mathematics and Financial Economics 5 *(Special Issue on Stochastic Financial Economics)*, 185–202.

[29] Evstigneev, I.V., Hens, T., and K.R. Schenk-Hoppé (2011) Survival and Evolutionary Stability of the Kelly Rule. In: *The Kelly Capital Growth Investment Criterion: Theory and Practice* (MacLean, L.C., Thorp, E.O., and W.T. Ziemba, eds.), pp. 273–284, World Scientific.

[30] Evstigneev, I.V., Schenk-Hoppé, K.R., and W.T. Ziemba (2013) Preface to the Special Issue *Behavioral and Evolutionary Finance*, Annals of Finance 9, 115–119.

[31] Friedman, M. and L.J. Savage (1948) The utility analysis of choices involving risk. Journal of Political Economy 56, 279–304.

[32] Gale, D. (1967) On Optimal Development in a Multi-sector Economy. Review of Economic Studies 34, 1–18.

[33] Grandmont, J.-M., ed. (1988) Temporary Equilibrium, Academic Press.

[34] Grandmont, J.-M., and W. Hildenbrand (1974) Stochastic Processes of Temporary Equilibria. Journal of Mathematical Economics 1, 247–277.

[35] Hens, T., and K.R. Schenk-Hoppé (2005) Evolutionary Stability of Portfolio Rules in Incomplete Markets. Journal of Mathematical Economics 41, 43–66.

[36] Hens, T., and K.R. Schenk-Hoppé, eds. (2009) Handbook of Financial Markets: Dynamics and Evolution. Volume in the Handbooks in Finance series (Ziemba, W.T., ed.), North-Holland.

[37] Hofbauer, J., and K. Sigmund (1998) Evolutionary Games and Population Dynamics. Cambridge University Press.

[38] Kahneman, D., and A. Tversky (1979) Prospect Theory: An Analysis of Decision under Risk. Econometrica 47, 263–292.

[39] Kelly, J.L. (1956) A New Interpretation of Information Rate. Bell System Technical Journal 35, 917–926.

[40] Kojima, F. (2006) Stability and Instability of the Unbeatable Strategy in Dynamic Processes. International Journal of Economic Theory 2, 41–53.

[41] Latané, H. (1959) Criteria for Choice Among Risky Ventures. Journal of Political Economy 67, 144–155.

[42] Levy, H., De Giorgi, E., and T. Hens (2012) Two Paradigms and Nobel Prizes in Economics: a Contradiction or Coexistence? European Financial Management 18, 163–182.

[43] MacLean, L.C., Thorp, E.O., and W.T. Ziemba, eds., (2011) The Kelly Capital Growth Investment Criterion: Theory and Practice, World Scientific.

[44] Magill, M., and M. Quinzii (2002) Theory of Incomplete Markets, Volume 1. MIT Press.

[45] Marshall, A. (1920) Principles of Economics, Macmillan.

[46] Maynard Smith, J., and G.R. Price (1973) The Logic of Animal Conflict. Nature 246, 15–18.

[47] Maynard Smith, J. (1982) Evolution and the Theory of Games. Cambridge University Press.

[48] McKenzie, L.W. (1986) Optimal Economic Growth, Turnpike Theorems and Comparative Dynamics. In: *Handbook of Mathematical Economics III* (Arrow, K.J., and M.D. Intrilligator, eds.), pp. 1281–1355, North-Holland.

[49] Mehra, R. and E.C. Prescott (1985) The equity premium: A puzzle. Journal of Monetary Economics 15, 145–161.

[50] Nikaido, H. (1968) Convex Structures and Economic Theory, Academic Press.

[51] Palczewski, J. and K.R. Schenk-Hoppé (2010) From Discrete to Continuous Time Evolutionary Finance. Journal of Economic Dynamics and Control 34, 913–931.

[52] Palczewski, J. and K.R. Schenk-Hoppé (2010) Market Selection of Constant Proportions Investment Strategies in Continuous Time. Journal of Mathematical Economics 46, 248–266.

[53] Radner, R. (1995) Equilibrium under Uncertainty. In: *Handbook of Mathematical Economics, Vol. II* (Arrow, K.J., and M.D. Intriligator, eds.) Elsevier.

[54] Samuelson, P. (1947) Foundations of Economic Analysis. Harvard University Press.

[55] Schaffer, M.E. (1988) Evolutionarily Stable Strategies for a Finite Population and a Variable Contest Size. Journal of Theoretical Biology 132, 469–478.

[56] Schlicht, E. (1985) Isolation and Aggregation in Economics. Springer Verlag.

[57] Shapley, L.S. (1953) Stochastic Games. Proceedings of the National Academy of Sciences (PNAS) 39, 1095–1100.

[58] Shiller, R.J. (2015) Irrational Exuberance (3rd ed.). Princeton University Press.

[59] Thorp, E.O. (1971) Portfolio Choice and the Kelly Criterion. In: *Stochastic Models in Finance* (Ziemba, W.T., and R.G. Vickson, eds.), pp. 599–619, Academic Press.

[60] Weibull, J.W. (1995) Evolutionary Game Theory. MIT Press.

[61] Zhou, X.Y. (2010) Mathematicalising Behavioural Finance. Proceedings of the International Congress of Mathematicians, Hyderabad, India.