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Unbeatable Strategies

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Abstract. The paper analyzes the notion of an unbeatable strategy as a game solution concept. A general framework (game with relative preferences) suitable for the analysis of this concept is proposed. Basic facts regarding unbeatable strategies are presented and a number of examples and applications considered.

1 Introduction

Nowadays Nash equilibrium is the most common solution concept in game theory. However, a century ago, when the discipline was in its infancy, the term "solving a game" was understood quite differently. The focus was not on finding a strategy profile that would equilibrate conflicting interests of the players. The main goal was to find, if possible, an individual strategy enabling the player to win (or at least not to lose) the game, in other words, to construct an *unbeatable strategy*. This question was considered in the paper by Bouton (1901-02), apparently the earliest mathematical paper in the field. Borel (1921) formulated in his study the general problem "to investigate whether it is possible to determine a method of play better than all others; i.e., one that gives the player who adopts it a superiority over every player who does not adopt it". When developing this idea, Borel introduced a famous class of games, that later received the name "Colonel Blotto games" (Borel 1921).

It should be noted that the problem of constructing unbeatable strategies turned out to be unsolvable for the vast majority of mind games of common interest (such as chess). What mathematicians could achieve, at most, was to prove that games in certain classes were determinate. A game is called *determinate* if at least one of the players has an unbeatable strategy.

Problems related to the determinacy of chess were considered in the paper by Zermelo (1913). Although this result is traditionally referred to as Zermelo's theorem,

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the determinacy of chess was apparently established for the first time by Kalmár (1929); for a discussion of the history of this question see Schwalbe and Walker (2001).

A deep mathematical analysis of the determinacy of infinite win-or-lose games of complete information was initiated by Gale and Stewart (1953). This line of studies has led to remarkable achievements in set theory and topology. The highlight in the field was "Martin's determinacy axiom" and a proof of its independence of the Zermelo–Fraenkel axioms of set theory (Martin 1975). For comprehensive surveys of research in this area see Telgársky (1987) and Kehris (1995); for reviews of topics related to unbeatable strategies in combinatorial game theory see Berlekamp et al. (1982).

However, the above achievements had for the most part purely theoretical value, having nothing to do with real-life applications. They dealt with elegant games created in the minds of mathematicians. A classical example is the Bouton's (1902) game "Nim", a theory of which was developed in his pioneering paper. Therefore new, applications-oriented solution concepts based on the idea of equilibrium (von Neumann 1928, Nash 1950) came to the fore. They became central in game-theoretic analysis, especially, in economics.

It should be noted that for symmetric games, unbeatable strategies, if they exist, are the same for both players and form a saddle point for the *associated zero-sum game* (see Section 3 below) whose payoffs are defined as differences between the payoffs of the players in the original game. This might be the reason why the topic of unbeatable strategies was "absorbed" by the analysis of saddle points and re-emerged only years after the seminal von Neumann 1928 paper. In the 1950s, when game theory started developing primarily as a mathematical framework of economic modeling, non-zero sum N-player games started prevailing in the field, and the notion of Nash equilibrium became central to the area.

The concept of an unbeatable strategy as such emerged again in theoretical biology and served as a starting point for the development of evolutionary game theory (EGT). Hamilton (1967) used this notion, and the term "unbeatable strategy"—without a rigorous formalization—in his paper on the analysis of sex ratios in populations of some species, which turned out to be extremely influential. Maynard Smith and Price (1973) formalized Hamilton's idea, but at the same time somewhat changed its content. The notion they introduced, usually referred to as an evolutionary stable strategy (ESS), should be called, more precisely, a *conditionally unbeatable* strategy. It it is indeed unbeatable, but only if the rival is "weak enough." In the context of evolutionary biology, ESS is a strategy which cannot be beaten if the fraction of the rivals (mutants) in the population is sufficiently small. This definition requires the population to be infinite, since one has to speak of its arbitrarily small fractions. Versions of ESS applicable to finite populations were suggested by Schaffer (1988, 1989). Schaffer's notions of ESS—there are two of these—are also in a sense conditionally unbeatable strategies. The first requires that the population contains only *one* mutant, the second assumes that there are several *identical* mutants.

It is not surprising that an unbeatable strategy, rather than a Nash equilibrium, turned out to be a key idea that fitted ideally the purposes of evolutionary modeling in biology. Nash equilibrium presumes full rationality of players, understood in terms of utility maximization, and their ability to coordinate their actions (or the presence of Harsanyi's "mediator") to establish an equilibrium, especially if it is non-unique. In a biological context such possibilities are absent, and moreover the role of individual utilities, always having a subjective nature, is played in EGT by a *fitness function*, an objective characteristic reflecting the survival rate in the natural selection process.

It is standard to present EGT models in conventional game-theoretic terms, with utilities/payoffs and Nash equilibrium, but this is just a matter of convenience, that makes it possible to employ the terminology and the results of conventional game theory. Moreover, EGT models are nearly exclusively symmetric, and as has been said above, the analysis of unbeatable strategies in the symmetric case boils down to the consideration of symmetric Nash equilibria (possessing some additional properties). At the same time, this kind of exposition, although convenient in some respects, might be misleading in others. In EGT, in contrast with conventional game theory, players do not select their strategies. Strategies are nothing but "genetic codes" of the players they have no influence on, while payoffs or utilities are not their individual characteristics (which are typically unobservable), but as has been noted, represent their fitness functions amenable to observations and statistical estimates.

The notion of ESS proposed by Maynard Smith and Price (1973) reigned in Evolutionary Game Theory for many years. An unconditional variant of ESS—fully corresponding to the Hamilton's idea of an unbeatable strategy[4]—was first revived in the context of economic applications of EGT in a remarkable paper by Kojima (2006), three decades after Maynard Smith and Price and four decades after Hamilton. The study of Kojima was motivated by the economic applications, where

[4]Hamilton (1967) did not give a rigorous general definition of an unbeatable strategy, using this notion in the specific context of that particular paper. In later papers (Hamilton and May, 1977; Comins et al., 1980), he used the notion of an ESS and emphasized its "combination of simplicity and generality". However, in Hamilton (1996), three decades later, he stated that in his 1967 paper he had in mind indeed a genuine notion of an unbeatable strategy, without the additional assumption of a small fraction of mutants in the population. For a discussion of the history of these ideas see Sigmund (2001).

the assumption of smallness of the population of "mutants" is obviously not realistic: a new technology or a new product can be thrown into the economy in any quantities.

In several years after Kojima's work, it was discovered (Amir et al. 2011, 2013) that the concept of an unbeatable strategy represents a very convenient and efficient tool in the analysis of financial market models combining evolutionary and behavioural principles—see the surveys in Evstigneev el al. (2015) and Holtfort (2019). This circumstance motivated us to undertake a systematic research on unbeatable strategies, which is conducted in the present paper.

The next section introduces a general framework (game with relative preferences) in which we examine unbeatable strategies. Section 2 analyzes in detail the case of a game with two players and cardinal preferences. In the remainder of the paper, various examples and applications are discussed. Some of these examples and applications are entirely new, some other are essentially known, and the novelty consists in their presentation from the perspective of unbeatable strategies.

2 Game with relative preferences

Game description. Consider a game of N players $i = 1, ..., N$ choosing their *strategies* x^i from some given sets X^i. Let $Z \subseteq X^1 \times ... \times X^N$ be the set of *admissible strategy profiles*. Let W^i be the set of *outcomes* of the game for player i. Denote by $w^i = w^i(x^1, ..., x^N) \in W^i$ the player i's outcome of the game that corresponds to the admissible strategy profile $(x^1, ..., x^N) \in Z$.

We would like to define the notion of an unbeatable strategy of some player i. To this end, we assume that for any pair of outcomes $w^i \in W^i,\ w^j \in W^j\ (j \neq i)$ a preference relation

$$w^i \succcurlyeq_{ij} w^j,\ w^i \in W^i,\ w^j \in W^j$$

is given. This preference relation is used to compare the game outcomes w^i and w^j of players i and j by estimating their relative performance. We do not impose any assumptions on the preference relation $\succcurlyeq_{ij}$; it is defined in terms of an arbitrary non-empty set $\{(w^i, w^j) \in W^i \times W^j : w^i \succcurlyeq_{ij} w^j\}$.

Definition. A strategy $\bar{x}$ of player i is called *unbeatable* if for any admissible strategy profile $(x^1, x^2, ..., x^N) \in Z$ with $x^i = \bar{x}$, we have

$$w^i(x^1, x^2, ..., x^N) \succcurlyeq_{ij} w^j(x^1, x^2, ..., x^N) \text{ for all } j \neq i. \tag{1}$$

According to this definition, player i adopting strategy $\bar{x}$ cannot be outperformed by any other player $j \neq i$ irrespective of what strategies player i's rivals $j \neq i$ use.

Reduction to the case of two players. Suppose we are interested in unbeatable strategies for player i in the above game. Then, as is easily seen, we can reduce the general game with N players to a game with two players, one of whom is player i while the other consists of the "team" $\{j : j \neq i\}$ of i's rivals. Indeed, define the strategy set $\hat{X}$ of the second player in the new game as

$$\hat{X} = \prod_{j \neq i} X^j,$$

(the set of the strategy profiles of the team of the i's rivals) and put

$$\hat{W} = \prod_{j \neq i} W^j.$$

Further, define

$$\hat{w}(\hat{x}) = (w^j(\hat{x}))_{j \neq i} \in \hat{W},$$

where

$$\hat{x} = x^{-i} = (x^j)_{j \neq i}.$$

The set $\hat{Z}$ of admissible strategy profiles in the new two-player game is as follows:

$$(x^i, \hat{x}) \in \hat{Z} \ \Leftrightarrow (x^i, x^{-i}) = (x^1, x^2, ..., x^N) \in Z.$$

The preference relation $w^i \hat{\succcurlyeq} \hat{w}$ used to compare the relative performance of player i and the group $\{j : j \neq i\}$ of her rivals means that

$$w^i \succcurlyeq_{ij} w^j \text{ for all } j \neq i,$$

where $\hat{w} = (w^j)_{j \neq i}$. Clearly $\bar{x}$ is an unbeatable strategy of player i in the original game if and only if $\bar{x}$ is an unbeatable strategy of player i in the new game.

Cardinal preferences (numeric measures of performance). Suppose that for each player i in the competitive game, a function $F_i(w^i)$, $w^i \in W^i$ is given. Assume that the preferences $\succcurlyeq_{ij}$ are defined in terms of the functions F_i as follows:

$$w^i \succcurlyeq_{ij} w^j \ \Leftrightarrow \ F_i(w^i) \geq F_j(w^j).$$

One can interpret $F_i(w^i)$ as the "score" assigned to player j if the outcome of the game for her is w^i.

Playing against the team of rivals. The performance of the team of the i's rivals $\hat{w} = (w^j)_{j \neq i}$ can be characterized by

$$\hat{F}(\hat{w}) = \max_{j \neq i} F_j(w^j).$$

For each admissible strategy profile $(x^1, x^2, ..., x^N)$, define

$$f_i(x^1, x^2, ..., x^N) = F_i(w^i(x^1, x^2, ..., x^N)).$$

Then a strategy $\bar{x}$ of player i is unbeatable if and only if for any admissible strategy profile $(x^1, x^2, ..., x^N)$ with $x^i = \bar{x}$, we have

$$f_i(x^1, x^2, ..., x^N) \geq f_j(x^1, x^2, ..., x^N), \;\; j \neq i,$$

or equivalently,

$$f_i(x^1, x^2, ..., x^N) \geq \max_{j \neq i} f_j(x^1, x^2, ..., x^N).$$

Symmetric N-player games. Let us say that the above game is *symmetric* if $X^1 = X^2 = ... = X^N$ and for every permutation $\pi(i)$ we have

$$f_i(x_1, ..., x_i, ..., x_N) = f_{\pi(i)}(x_{\pi(1)}, ..., x_{\pi(i)}, ..., x_{\pi(N)}), \tag{2}$$

in particular,

$$f_i(x_1, ..., x_i, ..., x_j, ..., x_N) = f_j(x_1, ..., x_j, ..., x_i, ..., x_N), \tag{3}$$

In the general case, if we wish to verify that $\bar{x}$ is an unbeatable strategy of some player, say player 1, then we need to check the validity of $N-1$ inequalities

$$f_1(\bar{x}, x_2, ..., x_N) \geq f_j(\bar{x}, x_2, ..., x_N) \text{ for all } j = 2, ..., N. \tag{4}$$

However, if the game is symmetric, it is sufficient to verify only one of these inequalities, for some particular j, say $j = 2$:

$$f_1(\bar{x}, x_2, ..., x_N) \geq f_2(\bar{x}, x_2, ..., x_N). \tag{5}$$

Proposition 2.1. *Inequalities (4) hold for all $x_2, ..., x_N$ if and only if inequality (5) holds for all $x_2, ..., x_N$.*

Proof. Assuming that (5) holds, consider any $j = 3, ..., N$, and observe that the inequality

$$f_1(\bar{x}, x_2, ..., x_N) \geq f_j(\bar{x}, x_2, ..., x_j, ..., x_N)$$

is equivalent to (5) because *(3)* implies

$$f_j(\bar{x}, x_2, ..., x_j, ..., x_N) = f_2(\bar{x}, x_j, ..., x_2, ..., x_N).$$

see (*2*). □

3 Two players, cardinal preferences

In this section, we examine in detail unbeatable strategies in games of two players with cardinal preferences specified by payoff functions. We repeat the general definitions formulated in the previous section in this particular case.

Unbeatable strategies in the present context. Consider a game of two players with strategy sets A, B and payoff functions $u(a,b)$, $v(a,b)$. Here, the payoff functions $u(a,b)$ and $v(a,b)$ are interpreted as *measures of performance* ("scores") of players 1 and 2. The goal of a player is to construct a strategy that would make it possible to outperform the rival (get a higher "score") whatever the rival's strategy might be.

Definition. Let us say that a strategy a^* of Player 1 is *unbeatable* if

$$u(a^*,b) \geq v(a^*,b) \text{ for any strategy } b \text{ of Player 2.} \tag{6}$$

Analogously, a strategy b^* of Player 2 is called *unbeatable* if

$$v(a,b^*) \geq u(a,b^*) \quad \text{for any strategy } a \text{ of Player 1.} \tag{7}$$

According to (6), if Player 1 uses the strategy a^*, then he will outperform Player 2, irrespective of her strategy b. Condition (7) expresses the analogous property of the strategy b^* of Player 2.

Determinacy. A game is said to be *determinate* if at least one of the players has an unbeatable strategy.

Unbeatable strategies in finite games. Consider a game of two players having finite strategy sets $A = \{a_1, ..., a_k\}$, $B = \{b_1, ..., b_l\}$ and payoffs $u_{ij} = u(a_i, b_j)$, $v_{ij} = v(a_i, b_j)$. According to the above definition, a strategy a_n of Player 1 is unbeatable if in the nth *row* of the payoff matrix, the payoffs of the first player exceed the corresponding payoffs of the second player: $u_{nj} \geq v_{nj}$. Analogously, a strategy b_m of Player 2 is unbeatable if in the mth *column* of the payoff matrix, the payoffs of the second player exceed the corresponding payoffs of the first player: $v_{im} \geq u_{im}$.

The associated zero-sum game. To analyze the concept of an unbeatable strategy we will associate with the original game a zero-sum game in which the strategy sets of players 1 and 2 are the same as above, A and B, while the payoff functions of players 1 and 2 are given by

$$f(a,b) = u(a,b) - v(a,b) \quad \text{and} \quad g(a,b) = -f(a,b).$$

Remark 3.1. If the original game is zero-sum, then $v(a,b) = -u(a,b)$, and so

$$f(a,b) = u(a,b) - v(a,b) = 2u(a,b),$$

which means that the associated zero-sum game is essentially equivalent to the original one.

Remark 3.2. If the original game is symmetric, i.e. $v(a,b) = u(b,a)$, then

$$f(a,b) = u(a,b) - v(a,b) = v(b,a) - u(b,a) = -f(b,a),$$

and consequently, the payoff function $f(a,b)$ in the associated zero-sum game is skew-symmetric:

$$f(a,b) = -f(b,a).$$

Thus $f(a,b) = g(b,a)$, which means that the associated zero-sum game is symmetric.

The associated game and unbeatable strategies. Recall that in the associated zero-sum game, the payoff functions are

$$f(a,b) = u(a,b) - v(a,b) \text{ and } g(a,b) = -f(a,b).$$

We can see that a strategy a^* of Player 1 is unbeatable if and only if $f(a^*, b) \geq 0$ for every strategy b of Player 2, or equivalently,

$$\min_b f(a^*, b) \geq 0. \tag{8}$$

Analogously, a strategy b^* of Player 2 is unbeatable if and only if $f(a, b^*) \leq 0$ for every strategy a of Player 1, or equivalently,

$$\max_a f(a, b^*) \leq 0. \tag{9}$$

Determinacy in terms of the associated game. It follows from (8) and (9) that Player 1 has an unbeatable strategy if and only if

$$\max_a \min_b f(a,b) \geq 0, \tag{10}$$

and Player 2 has an unbeatable strategy if and only if

$$\min_b \max_a f(a,b) \leq 0. \tag{11}$$

Thus the original game is determinate if and only if for the associated zero-sum game at least one of the conditions (10) and (11) holds.

Assumption 3.1. Here, as before, we assume that all the maximum and minimum values of all the functions we deal with are attained. This is so, e.g., in the following two cases: (i) the strategy sets A and B are finite; (ii) A and B are closed and bounded sets in R^N and the functions $u(a,b)$ and $v(a,b)$ are continuous.

Finite games. Consider a finite game with strategy sets $A = \{a_1, ..., a_k\}$, $B = \{b_1, ..., b_l\}$ and payoffs $u_{ij} = u(a_i, b_j)$, $v_{ij} = v(a_i, b_j)$. Denote by

$$f_{ij} = f(a_i, a_j) = u_{ij} - v_{ij},$$

the payoffs of Player 1 in the associated zero-sum game. In view of (8) and (9), we have the following: a strategy a_n of Player 1 is unbeatable if all the elements f_{nj} in the nth *row* of the payoff matrix (f_{ij}) are *non-negative*: $f_{nj} \geq 0$ for all j. Analogously, a strategy b_m of Player 2 is unbeatable if all the elements f_{im} in the mth *column* of the payoff matrix (f_{ij}) are *non-positive*: $f_{im} \leq 0$ for all i. Consequently, the original game is determinate if and only if the matrix (f_{ij}) has either a *non-negative row* or a *non-positive column.*

Example 3.1. Consider a game with the payoffs $u_{ij} = u(a_i, b_j)$, $v_{ij} = v(a_i, b_j)$ of players 1 and 2 and the associated zero sum game with the payoffs $f_{ij} = u_{ij} - v_{ij}$ of Player 1:

	b_1	b_2	b_3
a_1	$-3, -1$	$3, 1$	$2, 0$
a_2	$0, 1$	$4, 6$	$5, 4$
a_3	$1, 2$	$2, 4$	$3, 3$

		b_1	b_2	b_3	$\min_j f_{ij}$
	a_1	-2	2	2	-2
	a_2	-1	-2	1	-2
	a_3	-1	-2	0	-2
$\max_i f_{ij}$		-1	2	2	

We have

$$\min_j \max_i f_{ij} = -1, \ \max_i \min_j f_{ij} = -2.$$

• The matrix (f_{ij}) has a non-positive column, and so Player 2 has an unbeatable strategy.

• There are no non-negative rows in the matrix (f_{ij}), and so Player 1 does not have unbeatable strategies.

• The game is determinate but the associated zero sum game does not have a saddle point because $\max \min f_{ij} \neq \min \max f_{ij}$.

Saddle point implies determinacy.

Proposition 3.1. *If the associated zero-sum game has a saddle point, then the original game is determinate.*

Proof. If a saddle point exists, then, as we know, the value of the game

$$\bar{f} = \max_a \min_b f(a, b) = \min_b \max_a f(a, b).$$

is well defined. If $\bar{f} \geq 0$, then the condition

$$\max_a \min_b f(a,b) \geq 0 \tag{12}$$

holds, and so Player 1 has an unbeatable strategy. If $\bar{f} \leq 0$, then the condition

$$\min_b \max_a f(a,b) \leq 0 \tag{13}$$

holds, and so Player 2 has an unbeatable strategy.

The proof is complete.

Remark 3.3. The converse to Proposition 3.1 is not true. This follows from the above example. The game examined there is determinate but a saddle point in the associated zero-sum game does not exist ($\max\min f_{ij} \neq \min\max f_{ij}$).

Equivalence of determinacy and saddle point. Although the converse to Proposition 3.1 is not always true, it is true under one of the following additional assumptions regarding the original game: 1) the game is symmetric; 2) the game is a *win-or-lose game*.

Proposition 3.2. *Let the original game be symmetric. Then it is determinate if and only if the associated zero-sum game has a saddle point.*

Proof. In view of Proposition 3.1 it is sufficient to prove "only if": determinacy implies saddle point. Observe that in a symmetric game, if one of the players has an unbeatable strategy a^*, then the other player has the same unbeatable strategy. Let us show that a^*, a^* is a saddle point in the associated zero-sum game. For each a we have

$$f(a^*,a) = u(a^*,a) - v(a^*,a) \geq 0,\ \ f(a,a^*) = u(a,a^*) - v(a,a^*) \leq 0,$$

which implies that $f(a^*,a^*) = 0$. Therefore $f(a,a^*) \leq f(a^*,a^*) \leq f(a^*,a)$, which means that a^*, a^* is a saddle point of f.

Win-or-lose game is a game in which the payoffs $u(a,b)$ and $v(a,b)$ take on 2 values: 1 (win) and 0 (lose), and $u(a,b) = 1$ if and only if $v(a,b) = 0$.

Proposition 3.3. *A win-or-lose game is determinate if and only if the associated zero-sum game has a saddle point.*

Proof. "If" was proved in Proposition 3.1. Let us prove "only if". Suppose one of the players, say Player 1, has an unbeatable strategy a^*, i.e. $u(a^*,b) \geq v(a^*,b)$ for all b. This implies

$$u(a^*,b) = 1,\ \ v(a^*,b) = 0 \text{ for all } b. \tag{14}$$

Take any b^* in B. We claim that a^*, b^* is a saddle point. Indeed, $f(a^*,b^*) = u(a^*,b^*) - v(a^*,b^*) = 1 - 0 = 1$, and so $f(a,b^*) \leq f(a^*,b^*) = 1$ for all a. On the other hand,

$f(a^*, b^*) = f(a^*, b) = 1$ for all b by virtue of (14). Consequently, a^*, b^* is a saddle point.

Now assume that Player 2 has an unbeatable strategy b^*, i.e., $u(a, b^*) \leq v(a, b^*)$ for all a. This yields

$$u(a, b^*) = 0, \ v(a, b^*) = 1 \text{ for all } a. \tag{15}$$

Take any a^* in A. Let us show that a^*, b^* is a saddle point. Indeed, $f(a^*, b^*) = -1$, and so $f(a^*, b) \geq f(a^*, b^*) = -1$ for all b. Further, $f(a^*, b^*) = f(a, b^*) = -1$ for all a in view of (5). Thus, a^*, b^* is a saddle point.

On the existence of unbeatable strategies. We formulate a general existence theorem for unbeatable strategies. Let us introduce the following assumption:

$(*)$ For each strategy b of Player 2, there exists a strategy $a^*(b)$ of Player 1 satisfying

$$u(a^*(b), b) \geq v(a^*(b), b).$$

This assumption expresses the idea that for every strategy b of Player 2 there exists a strategy $a^*(b)$ of Player 1 which "beats" b, i.e. guarantees a better performance of Player 1 compared to Player 2.

Clearly, condition $(*)$ is necessary for the existence of an unbeatable strategy of Player 1. To prove the existence of such a strategy we need in fact to show (derive from $(*)$ under appropriate assumptions) that *the same* $a^*(b) = a^*$ with property $(*)$ can be selected *for all* b.

Assume that condition $(*)$ holds.

Theorem 3.1. *Let A be a convex, closed and bounded set in R^n and let B be a convex set in R^n. Let the functions $u(a, b)$ and $v(a, b)$ satisfy the following conditions: (i) $u(a, b)$ is continuous and concave in a and convex in b; (ii) $v(a, b)$ is continuous and convex in a and concave in b. Then Player 1 possesses an unbeatable strategy.*

Pyrrhic victory. Consider the game analyzed in the previous example:

	b_1	b_2	b_3
a_1	$-3, -1$	$3, 1$	$2, 0$
a_2	$0, 1$	$4, 6$	$5, 4$
a_3	$1, 2$	$2, 4$	$3, 3$

Clearly the strategy b_1 of Player 2 is unbeatable: it yields payoff greater than the payoff of Player 1, irrespective of his strategy. Thus b_1 is good in terms of the *relative* payoffs. However, in terms of the *absolute* payoffs, b_1 is the worst (strictly dominated by any other!) strategy of Player 2.

This seeming paradox demonstrates that the rationality in terms of a relative criterion may be wildly inconsistent with the rationality in terms of the absolute one. The strategy b_2 allows Player 2 to gain the victory over Player 1, but this is a Pyrrhic victory — a victory that is so devastating for the victor that it is tantamount to defeat. It is achieved as an outcome of a "*war of attrition*", at the expense of a dramatic reduction in Player 2's payoff, which is less, however, than the reduction in the payoff of Player 1.

Preference consistency for win-or-lose games. We point to some classes of games in which relative and absolute criteria of rationality are consistent. Recall that in a *win-or-lose game*, the payoffs $u(a, b)$ and $v(a, b)$ take on two values: 1 (win) and 0 (lose), and $u(a, b) = 1$ if and only if $v(a, b) = 0$.

By definition, a strategy a^* of Player 1 is unbeatable if and only if $u(a^*, b) \geq v(a^*, b)$ for all b, which in the present context means

$$u(a^*, b) = 1 \text{ and } v(a^*, b) = 0 \text{ for all } b. \tag{16}$$

This implies that

$$1 = u(a^*, b) \geq u(a, b) \text{ for all } a, b, \tag{17}$$

and so a^* is a weakly dominant strategy of Player 1. On the other hand, if a^* is weakly dominant and $u(a, b) = 1$ for at least one pair of strategies (a, b), then relations (17), and consequently, (16) are valid. This proves the following

Proposition 3.4. *In a win-or-lose game, an unbeatable strategy is weakly dominant. If at least one strategy of Player 1 can beat at least one strategy of Player 2, then any weakly dominant strategy is unbeatable.*

Preference consistency for symmetric zero-sum games. This is another class of games where the relative and absolute preferences of the players are consistent.

Proposition 3.5. *In a symmetric zero-sum game, a strategy is unbeatable if and only if it forms a symmetric NE.*

Proof. If a^* is an unbeatable strategy of Player 1, then

$$u(a^*, b) \geq v(a^*, b) = -u(a^*, b),$$

and so $u(a^*, b) \geq 0$. Further, we have

$$u(a^*, a^*) = -u(a^*, a^*) = 0.$$

Therefore

$$u(b, a^*) = v(a^*, b) = -u(a^*, b) \leq 0 = u(a^*, a^*),$$

and so (a^*, a^*) is a symmetric NE. Conversely, if (a^*, a^*) is a symmetric NE, then $u(b, a^*) \leq u(a^*, a^*)$, where $u(a^*, a^*) = -u(a^*, a^*) = 0$, which implies $u(a^*, b) = -u(b, a^*) \geq 0$ and

$$v(a^*, b) = u(b, a^*) \leq 0 \leq u(a^*, b).$$

Consequently, a^* is unbeatable.

4 Determinacy vs saddle point

Finite games with random payoffs. As we have seen above, for symmetric games the question of determinacy is equivalent to the question of the existence of a saddle point in the associated zero-sum game. If the game is not symmetric, this is not the case anymore. Moreover, it can be demonstrated that in a natural probabilistic sense, determinacy is "much more frequent" than saddle point. We will formalize this assertion and provide its proof—see Theorem 3 below.

Consider a zero-sum game with a finite set of strategies $i = 1, 2, ..., n$ and payoffs $g_{ij} = g(i, j)$ of the second player. As we have seen in Remark 1, in this case the associated zero-sum game has payoffs $\phi_{ij} = 2g_{ij}$, and obviously it possesses a saddle point if and only if the original game has a saddle point. Suppose that ϕ_{ij}, $i, j = 1, ..., n$, are independent identically distributed random variables with a uniform distribution on $[-1, 1]$. Denote by Δ_n the probability that the random game at hand is determinate and by Σ_n the probability that it has a saddle point.

Theorem 4.1. *We have*

$$\Delta_n = 2 - 2(1 - 2^{-n})^n, \tag{18}$$

$$\Sigma_n = n^2 2^{-n+1} \frac{(n-1)!}{(2n-1)!!}, \tag{19}$$

and

$$\Sigma_n / \Delta_n \to 0 \text{ as } n \to \infty \tag{20}$$

at an exponential rate.

The proof of Theorem 4.1 will be provided in the next three subsections.

How often is the game determinate? In other words, what is the probability Δ_n that in the random game described above, at least one of the players has an unbeatable strategy? By virtue of (10) and (11), Δ_n is the probability that

$$\alpha =: \max_i \min_j \phi_{ij} \geq 0 \text{ or } \beta = \min_j \max_i \phi_{ij} \leq 0, \tag{21}$$

i.e.

$$\Delta_n = P(A \cup B), \; A = \{\alpha \geq 0\}, \; B = \{\beta \leq 0\}.$$

We claim that $P(A \cap B) = 0$, i.e. the probability that both players have unbeatable strategies is equal to zero. Indeed, by virtue of Proposition 3, this happens if and only if the associated zero-sum game has a saddle point $\bar{x}, \bar{y}$ with $\phi(\bar{x}, \bar{y}) = 0$. This event, in turn, occurs if and only if $\alpha = \beta = 0$. But $P\{\alpha = 0\} = 0$ because the random variable α has a continuous distribution: its distribution function is differentiable, as it follows from formula (22) below. Therefore $\Delta_n = P(A \cup B) = P(A) + P(B)$.

For each $x \in [-1, 1]$, we have

$$P\{\alpha \leq x\} = P\{\min_j \phi_{ij} \leq x \text{ for each } i\} = P\{\min_j \phi_{ij} \leq x\}^n,$$

$$P\{\min_j \phi_{ij} \leq x\} = 1 - P\{\min_j \phi_{ij} > x\} = 1 - P\{\phi_{ij} > x \text{ for each } j\} =$$

$$1 - P\{\phi_{ij} > x\}^n = 1 - (\frac{1}{2} - \frac{x}{2})^n,$$

and so the distribution function of α is as follows:

$$P\{\alpha \leq x\} = [1 - (\frac{1}{2} - \frac{x}{2})^n]^n, \; x \in [-1, 1]. \tag{22}$$

From the above formulas, we obtain

$$P\{\alpha \geq 0\} = 1 - P\{\alpha < 0\} = 1 - P\{\alpha \leq 0\} = 1 - (1 - 2^{-n})^n.$$

To compute the probability $P\{\beta \leq 0\}$, we write

$$P\{\min_j \max_i \phi_{ij} > 0\} = P\{\max_i \phi_{ij} > 0 \text{ for each } j\} = P\{\max_i \phi_{ij} > 0\}^n,$$

$$P\{\max_i \phi_{ij} > 0\} = 1 - P\{\max_i \phi_{ij} \leq 0\} = 1 - P\{\phi_{ij} \leq 0\}^n = 1 - 2^{-n}$$

$$P\{\beta \leq 0\} = 1 - (1 - 2^{-n})^n.$$

Thus, we conclude that

$$\Delta_n = P\{\alpha \geq 0\} + P\{\beta \leq 0\} = 2 - 2(1 - 2^{-n})^n.$$

How often does the game have a saddle point? Let us compute the probability Σ_n of the existence of a saddle point in the game under consideration. We claim that Σ_n is equal to $n^2\bar{\Sigma}_n$, where $\bar{\Sigma}_n$ is the probability that the game has a saddle point at some particular pair of strategies $\bar{\imath}, \bar{\jmath}$. To show this it is sufficient to prove the following:

$$P\{A(i_0, j_0) \cap A(i_1, j_1)\} = 0 \text{ for any } (i_0, j_0) \neq (i_1, j_1),$$

where $A(i, j)$ is the the event that (i, j) is a saddle point. If both events $A(i_0, j_0)$ and $A(i_1, j_1)$ occur, then

$$\max_i \min_j \phi_{ij} = \min_j \phi_{i_0 j} = \min_j \phi_{i_1 j},$$

or in other words, $\psi_{i_0} = \psi_{i_1}$, where $\psi_i = \min_j \phi_{ij}$. The random variables $\psi_1, ..., \psi_n$ are independent and have the same continuous distribution. Consequently, the probability that their maximum is attained simultaneously at two different points i_0 and i_1 is equal to zero.

It remains to compute the probability $\bar{\Sigma}_n$. To this end we write

$$\bar{\Sigma}_n = P\{\phi_{i,\bar{\jmath}} \leq \phi_{\bar{\imath},\bar{\jmath}} \leq \phi_{\bar{\imath},j} \text{ for all } i, j\} =$$

$$\int_{-1}^{1} \frac{1}{2} P\{\phi_{i,\bar{\jmath}} \leq x \leq \phi_{\bar{\imath},j} \text{ for all } i \neq \bar{\imath}, j \neq \bar{\jmath} \; |\phi_{\bar{\imath},\bar{\jmath}} = x\} dx =$$

$$\frac{1}{2} \int_{-1}^{1} P\{\phi_{i,\bar{\jmath}} \leq x \leq \phi_{\bar{\imath},j} \text{ for all } i \neq \bar{\imath}, j \neq \bar{\jmath}\} dx =$$

$$\frac{1}{2} \int_{-1}^{1} P\{\phi_{i,\bar{\jmath}} \leq x \text{ for all } i \neq \bar{\imath}\} P\{x \;\leq \phi_{\bar{\imath},j} \text{ ,for all } j \neq \bar{\jmath}\} dx =$$

$$\frac{1}{2} \int_{-1}^{1} P\{\phi_{i,\bar{\jmath}} \leq x\}^{n-1} P\{x \;\leq \phi_{\bar{\imath},j}\}^{n-1} dx =$$

$$2^{-(2n-1)} \int_{-1}^{1} (1 + x)^{n-1} (1 - x)^{n-1} dx =$$

$$2^{-(2n-1)} \int_{-1}^{1} (1 - x^2)^{n-1} dx = 2^{-(2n-1)} \cdot 2 \cdot I_n = 2^{-2n+2} I_n,$$

where

$$I_n := \int_0^1 (1 - x^2)^{n-1} dx.$$

To compute I_n let us make the change of variables $x = \sin y$. Then we have $1 - x^2 = 1 - \sin^2 y = \cos^2 y$ and $dx = \cos y \; dy$. Therefore

$$I_n = \int_0^{\pi/2} \cos^{2(n-1)} y \cdot \cos y \; dy = \int_0^{\pi/2} \cos^{2n-1} y \; dy =$$

$$\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdot\ldots\cdot\frac{2}{3}=\frac{2^{n-1}(n-1)!}{(2n-1)!!}$$

and

$$\bar{\Sigma}_n=2^{-2n+2}\cdot 2^{n-1}\cdot\frac{(n-1)!}{(2n-1)!!}=2^{-n+1}\frac{(n-1)!}{(2n-1)!!},$$

which yields (19).

Determinacy occurs "much more often" than saddle point. To complete the proof of Theorem 3 it remains to prove (20). We first observe that

$$l:=\lim_{n\to\infty}\Delta_n/\frac{n}{2^{n-1}}=\lim_{n\to\infty}\frac{2^n-2^n(1-2^{-n})^n}{n}=1.$$

This follows from the fact that

$$l=\lim_{x\to 0+}x[2^{\frac{1}{x}}-2^{\frac{1}{x}}(1-2^{-\frac{1}{x}})^{\frac{1}{x}}]=1.$$

Thus in order to establish (20) we write:

$$\frac{n}{2^{n-1}}/\Sigma_n=\frac{n}{n^2 2^{-n+1}2^{n-1}}\frac{(2n-1)!!}{(n-1)!}=\frac{1}{n}\cdot\frac{(2n-1)!!}{(n-1)!}=$$

$$\frac{1}{n}\cdot\frac{(2n-2)+1}{(n-1)}\cdot\frac{(2n-4)+1}{(n-2)}\cdot\ldots\cdot\frac{4+1}{2}\cdot\frac{3}{1}=$$

$$\frac{1}{n}\cdot(2+\frac{1}{n-1})\cdot\ldots\cdot(2+\frac{1}{2})\cdot 3\geq\frac{3\cdot 2^{n-1}}{n}.$$

We can see that $\Delta_n/\Sigma_n\to\infty$ at an exponential rate, which completes the proof of Theorem 4.1. □

5 Unbeatable strategies in evolutionary game theory

Our next goal is to demonstrate applications of unbeatable strategies in Evolutionary Game Theory.

Population model. Members of a population of organisms (e.g. animals, human beings, plants, etc.) interact pairwise. Each organism can be of a certain type x. The set of possible types is X. There is a function $u(x,y)$, $x,y\in X$ (*fitness function*) that characterizes the ability of organisms to survive. If an organism is of a type x

and faces the probability distribution β of types y in the population, then its ability to survive is characterized by the expectation of $u(x, y)$ with respect to β.

In evolutionary biology, elements x in X might represent *genotypes* of species and $u(x, y)$ the (average) number of surviving offspring. In evolutionary economics, such models serve to describe interactions in large populations of economic agents. Types x can represent various characteristics of economic agents and/or patterns of their behaviour.

Symmetric game. With the given model, we associate a *symmetric* two-player game in which the payoff functions of the players are $u(x, y)$ and $v(x, y) = u(y, x)$, and their common strategy set is X. In this context, we will use the terms "types" and "strategies" interchangeably. The values of the fitness function $u(x, y)$ will be interpreted as "payoffs".

Consider the evolutionary population model (see, e.g., Weibull 1995). Let X be the set of possible types of interacting organisms/economic agents in the population and $u(x, y)$, $x, y \in X$, the fitness function. Let us say that a strategy x^* is *(strictly) unbeatable* if

$$(1-\varepsilon)u(x^*, x^*) + \varepsilon u(x^*, x) > \ (1-\varepsilon)u(x, x^*) + \varepsilon u(x, x) \tag{23}$$

for all $x \neq x^*$ and all $0 < \varepsilon < 1$. We will omit "strictly" in what follows.

Evolutionary interpretation of an unbeatable strategy. For an unbeatable strategy x^*, the inequality (23) must hold for all $x \neq x^*$ and all $0 < \varepsilon < 1$. For an ESS x^*, it must hold only for $\varepsilon > 0$ small enough, which means that "non-mutants" x^* outperform "mutants" x only if the fraction of the mutants is small enough. The definition of an unbeatable strategy requires that this should be true always, not only when the fraction of the mutants is sufficiently small.

Proposition 5.1. *A strategy x^* is unbeatable, i.e. (23) holds for all $0 < \varepsilon < 1$, if and only if for each $x \neq x^*$ at least one of the following conditions*

$$u(x, x^*) < u(x^*, x^*) \text{ and } u(x, x) \leq u(x^*, x), \tag{24}$$

$$u(x, x^*) \leq u(x^*, x^*) \text{ and } u(x, x) < u(x^*, x) \tag{25}$$

is fulfilled.

Proof. Suppose (24) holds. Multiply the first inequality in (24) by $1 - \varepsilon$, the second by ε, and add up. This will yield (23). The same argument shows that (25) implies (23).

Conversely, observe that inequality (23) holds for each $0 < \varepsilon < 1$ if and only if it holds as a non-strict inequality both for $\varepsilon = 0$ and $\varepsilon = 1$ and as a strict inequality in at least one of the two cases: $\varepsilon = 0$ and $\varepsilon = 1$ The former case corresponds to (24) and the latter to (25). □

Remark 5.1. We comment on the meaning of conditions (24) and (25) in terms of dominance of the strategy x^* in the two-player two-strategy game

$u(x^*, x^*)$	$u(x^*, x)$
$u(x, x^*)$	$u(x, x)$

.

Conditions (24) and (25) mean that in this game the strategy x^* is *dominant* in the following sense: the inequalities

$$u(x, x^*) \leq u(x^*, x^*), \; u(x, x) \leq u(x^*, x)$$

hold and at least one of them is strict. Recall that if both inequalities are strict, then the strategy x^* is called strictly dominant. If both inequalities hold but none of them is assumed to be strict, then the strategy x^* is called weakly dominant. The above dominance property is "intermediate" between the last two.

Remark 5.2. We compare (24) and (25) with the conditions defining ESS: for each $x \neq x^*$, we have either

$$u(x, x^*) < u(x^*, x^*) \tag{26}$$

or

$$u(x, x^*) = u(x^*, x^*) \text{ and } u(x, x) < u(x^*, x). \tag{27}$$

Note that the assertion that at least one of the conditions (24) and (25) holds is equivalent to the assertion that for each $x \neq x^*$, one (and only one) of the following two requirements is fulfilled:

(I) $u(x, x^*) < u(x^*, x^*)$ and $u(x, x) \leq u(x^*, x)$,

or

(II) $u(x, x^*) = u(x^*, x^*)$ and $u(x, x) < u(x^*, x)$.

This is clear because the inequality $u(x, x^*) \leq u(x^*, x^*)$ involved in (25) can hold either as a strict inequality, and then we have (I), or as equality, which leads to (II).

Remarkably, condition (II) coincides with property (27) in the definition of ESS, but condition (I) contains together with the strict equilibrium property $u(x, x^*) < u(x^*, x^*)$ stated in (26) the additional requirement $u(x, x) \leq u(x^*, x)$.

Mixed strategies and simple games. Let X be the set of mixed strategies in a symmetric game with two strategies a_1, a_2 and the payoffs $u_{ij} = u(a_i, a_j)$ of the first player. Unbeatable strategies (as well as ESS and mixed-strategy NE) are defined in terms of the differences $u(\alpha, \beta) - u(\beta, \beta)$, where $\alpha = (p, 1-p)$ and $\beta = (q, 1-q)$ are mixed strategies. Therefore unbeatable strategies are the same for the original game and the reduced simple game

$u_1 = u_{11} - u_{21}$	0
0	$u_2 = u_{22} - u_{12}$

In the analysis of this game, we will assume, as before, that $u_1 \neq 0$ and $u_2 \neq 0$.

We characterize those mixed strategies $\beta = (q, 1-q)$ which form an unbeatable strategy in the game at hand, i.e. satisfy for all $\alpha \neq \beta$ conditions (I) or (II):
(I) $u(\alpha, \beta) < u(\beta, \beta)$ and $u(\alpha, \alpha) \leq u(\beta, \alpha)$,
(II) $u(\alpha, \beta) = u(\beta, \beta)$ and $u(\alpha, \alpha) < u(\beta, \alpha)$.
We know the structure of ESS:

		ESS	$\beta = (q, 1-q)$
Case 1	$u_1 < 0, u_2 < 0$	one	$q = q^*$, $q^* = \dfrac{u_2}{u_1 + u_2}$, $1 - q^* = \dfrac{u_1}{u_1 + u_2}$,
Case 2	$u_1 > 0, u_2 > 0$	two	$q = 0,\ 1,$
Case 3	$u_1 < 0, u_2 > 0$	one	$q = 0,$
Case 4	$u_1 > 0, u_2 < 0$	one	$q = 1.$

Which of these ESS are unbeatable?

Case 1. The mixed strategy $\beta = (q, 1-q)$ with $q = q^*$ is an unbeatable strategy. Indeed, we know that $u(\alpha, \beta) = u(\beta, \beta)$ and $u(\alpha, \alpha) < u(\beta, \alpha)$ for all $a \neq \beta$, which is precisely condition (II).

In all the other cases (cases 2-4), ESS are strict and so we have only to check the second inequality in
(I) $u(\alpha, \beta) < u(\beta, \beta)$ and $u(\alpha, \alpha) \leq u(\beta, \alpha)$,
which can be written as

$$u(\alpha, \alpha) = p^2 u_1 + (1-p)^2 u_2 \leq pqu_1 + (1-p)(1-q)u_2 = u(\beta, \alpha). \tag{28}$$

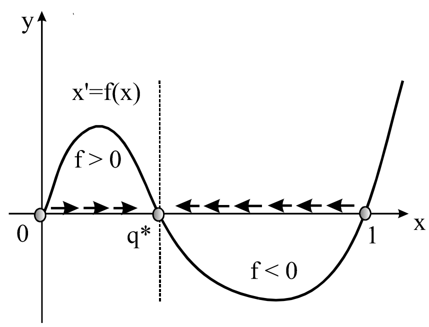
Case 2: neither $q = 0$, nor $q = 1$ are unbeatable. Indeed, if $q = 0$, then (23) becomes $p^2 u_1 + (1-p)^2 u_2 \leq (1-p)u_2$, which is not true for $p = 1$. If $q = 1$, then (23) yields $p^2 u_1 + (1-p)^2 u_2 \leq pu_1$, which is wrong for $p = 0$.

Case 3: $q = 0$ is unbeatable because $p^2 u_1 + (1-p)^2 u_2 \leq (1-p)^2 u_2 \leq (1-p)u_2$.
Case 4: $q = 1$ is unbeatable because $p^2 u_1 + (1-p)^2 u_2 \leq p^2 u_1 \leq pu_1$.
We summarize the results obtained in the following table:

		unbeatable strategies	$(q, 1-q)$
Case 1	$u_1 < 0, u_2 < 0$	one	$q = q^*$, $q^* = \dfrac{u_2}{u_1 + u_2}$,
Case 2	$u_1 > 0, u_2 > 0$	no unbeatable strategies	
Case 3	$u_1 < 0, u_2 > 0$	one	$q = 0$,
Case 4	$u_1 > 0, u_2 < 0$	one	$q = 1$.

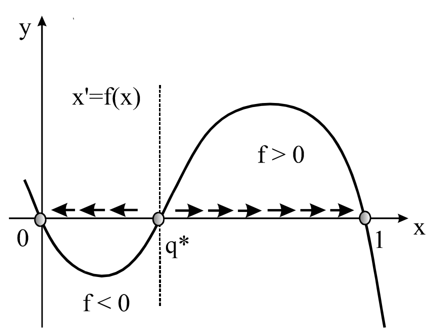
Unbeatable strategies and replicator dynamics. We will show that for the replicator dynamics process those ESSS which correspond to unbeatable strategies form globally evolutionary stable steady states (GESSS). By definition, a steady state x of a dynamical system is *globally stable* if trajectories of the system starting from *any* initial point in the domain of the system (and not only from points in some sufficiently small neighborhood of x) converge to x.

Case 1: $u_1 < 0$, $u_2 < 0$. As we have seen, $q^* = u_1/(u_1 + u_2)$ corresponds to the unique unbeatable strategy. The dynamics of the RD process in this case is as follows:



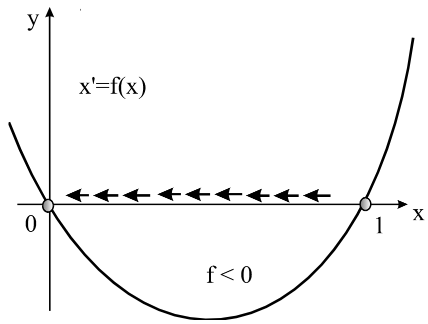
The RD process converges to q^* starting from *any* initial point x in the interval $0 < x < 1$, and not only from points sufficiently close to q^*. Consequently, q^* is GESSS.

Case 2: $u_1 > 0, u_2 > 0$. The dynamics of the RD process is illustrated in the following diagram:



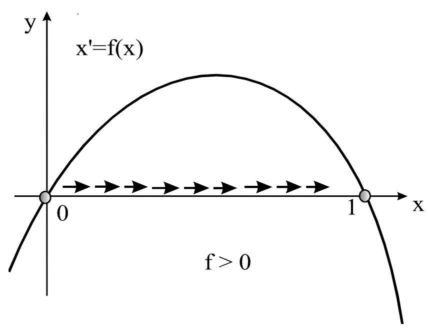
There are two ESS: $q = 0$ and $q = 1$, but *none* of them is GESSS. Convergence to 1 takes place only for trajectories of the RD process starting from initial states $q^* < x < 1$ and convergence to 0 takes place only for trajectories starting from $0 < x < q^*$ (recall that $q^* = u_2/(u_1 + u_2)$).

Case 3: $u_1 < 0$, $u_2 > 0$. The dynamics of the RD process is shown in the following diagram:



The ESS $q = 0$ is globally evolutionary stable since the RD process starting from *every* initial state $0 < x < 1$ converges to 0.

Case 4: $u_1 > 0$ and $u_2 < 0$. The function $f(x)$ is strictly positive for $0 < x < 1$, and therefore the RD process starting from *every* point $0 < x < 1$ converges to 1.



In all the cases considered, *the globally evolutionary stable steady states of the replicator dynamics process correspond to the unbeatable strategies of the underlying game!*

6 From competition to contest in Cournot duopoly: Bertrand's paradox

Contest vs. Competition. A firm producing a homogeneous good owns two plants. The plants are run by two managers, modeled as players in the Cournot duopoly game. The goal of the firm, serving the whole market, is to maximize profits. To achieve this goal it contemplates an incentive scheme for the managers. Rather than allocating to them some fixed percentage of profits, the firm sets up a contest. A fixed prize/bonus $B > 0$ is awarded to that manager who succeeds in getting a higher profit than the other. If their profits are equal, they share the award equally: each gets $B/2$. A paradoxical outcome of this contest is that of perfect competition. It turns out to be rational for both players to follow strategies leading to minimum prices and zero profits, which is disastrous for the profit maximizing firm.

The model. The vehicle for our analysis is the classical Cournot duopoly game with a homogeneous good, nonlinear inverse demand and symmetric linear costs. There are two firms $i = 1, 2$ producing quantities $q_i \geq 0$ of a homogeneous good. Firm i's profit is

$$\pi_i(q_1, q_2) = q_i P(Q) - cq_i \; [i = 1, 2, \; Q = q_1 + q_2],$$

where $c > 0$ is the marginal cost of production. The function $P(Q)$ specifies the market clearing price (inverse demand), depending on the total quantity produced $Q = q_1 + q_2$. In this game, the players $i = 1, 2$ have the payoff functions $\pi_i(q_1, q_2)$ and the common strategy set consisting of all non-negative numbers q_i. The game is symmetric: $\pi_1(q_1, q_2) = \pi_2(q_2, q_1)$.

Suppose there exists a strategy q^* that would guarantee a positive award (B or $B/2$) for player 1 irrespective of the strategy of player 2. Such a strategy must satisfy

$$\pi_1(q^*, q) \geq \pi_2(q^*, q) \tag{29}$$

for all $q \geq 0$. Let us call such a strategy *unbeatable.* It allows player 1 to outperform the rival (or at least to achieve the same result) in terms of profit maximization irrespective of the rival's strategy. Since the game is symmetric, the sets of unbeatable strategies for players 1 and 2 coincide.

If a strategy q^* with property (29) exists, it will be rational for the participants of the contest to select it. Indeed, q^* will guarantee a bonus of at least $B/2$, whatever the rival undertakes. If some strategy of the first player is not unbeatable, then the second player can act so as to get a strictly higher profit than the first, in which case the first will get no bonus. Thus those and only those strategies can be viewed as solutions to the contest game that are unbeatable.

A key result. We introduce the following hypothesis:

(C) There exists $\bar{Q} > 0$ such that (i) $P(\bar{Q}) = c$, (ii) $P(Q) < c$ for each $Q > \bar{Q}$, and (iii) $P(Q) > c$ for each $Q < \bar{Q}$.

Proposition 6.1. *Under condition (C), there exists a unique unbeatable strategy* $q^* = \bar{Q}/2$, *and we have*

$$\pi_1(q^*, q) > \pi_2(q^*, q) \text{ for } q \neq q^*. \tag{30}$$

When both players use (the same and unique) unbeatable strategy $\bar{Q}/2$, this leads to the price $P(\bar{Q}) = c$ and zero profits $\pi_i(q^*, q^*) = 0$. An outcome of this kind is characteristic for perfect competition. Typically, perfect competition emerges when the number of firms tends to infinity. In the present model, it takes place with only two firms. An analogous phenomenon can be observed in Bertrand's duopoly (Bertrand's paradox).

Clearly condition (C) is satisfied for the Cournot duopoly with linear inverse demand

$$P(Q) = \begin{cases} P(Q) = 1 - Q, & \text{if } 0 \leq Q \leq 1, \\ P(Q) = 0, & \text{if } Q > 1, \end{cases}$$

and a production cost $0 < c < 1$. In this case, $\bar{Q} = 1 - c$. Thus, by virtue of Theorem 1, $q^* = (1-c)/2$ is the unique unbeatable strategy.

Proof of Proposition 6.1. Define

$$f(q^*, q) = \pi_1(q^*, q) - \pi_2(q^*, q) = (q^* - q)[P(q^* + q) - c].$$

Let $q^* = \bar{Q}/2$. If $q > q^*$, then $q^* + q > \bar{Q}$, and by virtue of (C), $P(q^* + q) - c < 0$. Since $q^* - q < 0$, we have $f(q^*, q) > 0$. If $q < q^*$, then $q^* + q < \bar{Q}$, and according to (C), $P(q^* + q) - c > 0$. Together with the inequality $q^* - q > 0$, this yields $f(q^*, q) > 0$. If $q = q^*$, then $f(q^*, q) = 0$. Consequently, $q^* = \bar{Q}/2$ is an unbeatable strategy, and inequality (30) holds.

Let us prove that if q^* is an unbeatable strategy, then $q^* = \bar{Q}/2$. Suppose $q^* \neq \bar{Q}/2$ is an unbeatable strategy. Put $q = \bar{Q}/2$. If $q^* < \bar{Q}/2$, then $q^* - q < 0$ and $q^* + q < \bar{Q}$, which yields $P(q^* + q) - c > 0$ and so $f(q^*, q) < 0$. If $q^* > \bar{Q}/2$, then $q^* - q > 0$ and $q^* + q > \bar{Q}$, which yields $P(q^* + q) - c < 0$ and so $f(q^*, q) < 0$. We assumed that $q^* \neq \bar{Q}/2$ and arrived at a contradiction. □

Oligopoly contest. Consider an analogous contest with N players $i = 1, 2, ..., N$. Each of the players $i = 1, 2, ..., N$ selects a strategy $q_i \geq 0$ and gets the payoff

$$\pi_i(q_1, ..., q_N) = q_i P(Q) - cq_i \ ,$$

where $Q = \sum_{i=1}^{N} q_i$. If K of them get the highest payoff, they share the bonus B equally, so that each gets B/K. A solution to this contest game is an unbeatable strategy q^*, for which, by definition, the following inequalities hold:

$$q^* P(q^* + \sum_{j=2}^{N} q_j) - cq^* \geq q_i P(q^* + \sum_{j=2}^{N} q_j) - cq_i, \ q_i \geq 0, \ i = 2, ..., N. \tag{31}$$

Proposition 6.2. *There are no unbeatable strategies in the oligopoly contest with $N > 2$ players.*

Proof. If $q_3 = ... = q_N = 0$, the inequalities in (31) impose on q^* the same conditions as in the game with two players analyzed above. As Proposition 1 shows, there can be only one strategy satisfying these conditions: $q^* = \bar{Q}/2$. Thus if an unbeatable strategy in the oligopoly contest exists, it must be $q^* = \bar{Q}/2$. Then, in view of (31), the following inequality must hold

$$\frac{\bar{Q}}{2}[P(\frac{\bar{Q}}{2} + q + q') - c] \geq q[P(\frac{\bar{Q}}{2} + q + q') - c],$$

where q and $q' := \sum_{j=3}^{N} q_j$ are any non-negative numbers. By setting $q = 0$ and $q' = \bar{Q}$, we obtain the inequality

$$\frac{\bar{Q}}{2}[P(\frac{3\bar{Q}}{2}) - c] \geq 0,$$

which cannot hold by virtue of the assumption (C). The contradiction obtained proves the proposition. □

7 From contest to competition in Cournot duopoly: Transformational recession

Cournot duopoly with differentiated products. Assume that two firms produce two different goods. Firm 1 produces quantity $q_1 \geq 0$ of good 1, and firm 2 produces quantity $q_2 \geq 0$ of good 2. The inverse demand functions for the two goods are

$$P_1(q_1, q_2) = 1 - q_1 - bq_2,\ P_2(q_1, q_2) = 1 - q_2 - bq_1.$$

The number $b \in (0, 1]$ reflects the extent to which the firms' products are substitutes to each other. The payoff functions (profits) are

$$\pi_i(q_1, q_2) = q_i P_i(q_1, q_2) - cq_i,\ i = 1, 2\ ,$$

where $0 < c < 1$ is the marginal cost of production. The case $b = 1$ corresponds to the case of a homogeneous good.

Nash equilibrium. Let us find NE (q_1^*, q_2^*). The quantity q_1^* has to maximize

$$q_1(1 - q_1 - bq_2^*) - cq_1 \text{ over } q_1 \geq 0.$$

The first order optimality condition

$$-2q_1^* + 1 - c - bq_2^* = 0$$

gives $q_1^* = (1 - c - bq_2^*)/2$. Analogously, for firm 2, we find $q_2^* = (1 - c - bq_1^*)/2$.

Solving the system of equations

$$q_1^* = (1 - c - bq_2^*)/2, \; q_2^* = (1 - c - bq_1^*)/2,$$

we obtain that the *equilibrium quantities* $q^* := q_1^* = q_2^*$ satisfy$q^* = (1 - c - bq^*)/2$.

Thus

$$q^* = q_1^* = q_2^* = \frac{1-c}{2+b}.$$

The *equilibrium profits* are as follows:

$$\pi^* := \pi_i(q^*, q^*) = \frac{(1-c)^2}{(2+b)^2}.$$

Indeed, we have

$$\pi_i(q^*, q^*) = (q^* - c)P_i(q^*, q^*) = q^*(1 - q^* - bq^* - c) =$$

$$q^*[-q^*(b+1) + 1 - c] = \frac{1-c}{2+b} \cdot [-\frac{1-c}{2+b}(b+1) + 1 - c] =$$

$$\frac{(1-c)^2}{2+b}[-\frac{b+1}{2+b} + 1] = \frac{(1-c)^2}{2+b}\frac{1}{2+b} = \frac{(1-c)^2}{(2+b)^2}.$$

Contest. In the contest, each of the players strives to outperform the other (in terms of the profit obtained) in order to win a fixed prize/bonus. The solution to this contest for each player is an *unbeatable strategy*, that makes it possible to always outperform (not necessarily strictly) the rival. If both players achieve the same results, they share the bonus evenly. By definition, an unbeatable strategy $q_1 = \bar{q}$ of player 1 satisfies

$$\bar{q}(1 - \bar{q} - bq_2) - c\bar{q} \geq q_2(1 - q_2 - b\bar{q}) - cq_2.$$

This can be equivalently written

$$\bar{q}(1 - \bar{q}) - c\bar{q} \geq q_2(1 - q_2) - cq_2.$$

Thus $\bar{q}$ maximizes

$$q(1 - q) - cq = -q^2 + q(1 - c),$$

consequently, the *unbeatable strategy of player 1 (and by symmetry, of player 2)* in the contest is

$$\bar{q} = \frac{1-c}{2}.$$

The *profit resulting from the contest*, when both players use unbeatable strategies, is equal to

$$\bar{\pi} := \frac{(1-c)^2(1-b)}{4}.$$

Indeed, we have

$$\bar{\pi} = \bar{q}(1-\bar{q}-b\bar{q}-c) = \frac{1-c}{2}\cdot[1-c-(b+1)\frac{(1-c)}{2}] =$$

$$\frac{(1-c)^2}{2}[1-b/2-1/2] = \frac{(1-c)^2(1-b)}{4}.$$

Remark 7.1. *Total equilibrium output*

$$2q^* = \frac{2(1-c)}{(2+b)}$$

is always not greater than the total contest output

$$2\bar{q} = 1-c,$$

when both players employ unbeatable strategies. In the case of a homogeneous good $(b=1)$, we have

$$2q^* = \frac{2(1-c)}{(2+b)} = \frac{2(1-c)}{3}.$$

Thus, when passing from the contest to competition we observe, paradoxically, a decline in the production output: a *transitional recession*. Why paradoxically? Because it is commonly accepted that competition increases efficiency.

Note that the depth of the recession depends on the degree of substitutability, b, of the goods produced by firms 1 and 2. It increases when b increases. The maximum depth of the recession is observed when $b=1$, in the case of a homogeneous good. In this case production falls by 1/3 (see the last two formulas).

Remark 7.2. *The equilibrium profit*

$$\pi^* = \frac{(1-c)^2}{(2+b)^2}$$

is always not less than the contest profit

$$\bar{\pi} = \frac{(1-c)^2(1-b)}{4},$$

when both players use unbeatable strategies. Indeed, the inequality $\pi^* \geq \bar{\pi}$ is equivalent to

$$(2+b)^2(1-b) \leq 4.$$

The last relation is true because the derivative of the function $\gamma(b) := (2+b)^2(1-b)$, which is equal to

$$\gamma'(b) = 2(2+b)(1-b) - (2+b)^2 = 4 + 2b - 4b - 2b^2 - 4 - 4b - b^2 = -6b - 3b^2,$$

is negative on $[0,1]$, and $\gamma(0) = 4$.

It is important to note that the outcome of the contest in the model with a homogeneous good (i.e. when $b = 1$), is that of *perfect competition:* the profit $\bar{\pi}$ is equal to zero. In this case, in the course of transition from contest to competition, the profit increases from zero to $\pi^* = (1-c)^2/4$. If $b < 1$, the profit increases as well, but not that drastically.

Remark 7.3. The term "transformational recession" refers to the decline of the production output observed in the 1990s in most of the post-socialist countries in the course of passing from planned to decentralized economy (see, e.g., Polterovich 1993, Alexeev and Weber 2013). One of the factors that hypothetically influenced the recession was an abrupt transition from non-profit, contest-type incentives that played a key role in centralized economies ("socialist competition", see, e.g., Jeffries 1989, Kornai 1992, Ilič and Miklóssy 2014) by profit maximization in a still essentially non-market environment. The analysis conducted in this section might be regarded as a toy model supporting this hypothesis.

8 Dynamic games with alternating moves

This section develops a mathematical framework for dynamic two-player games of complete information with an alternating order of the players' moves. The framework comprises many famous and popular mind games such as chess. The main result is a proof of the (strong) determinacy of games of this kind.

Dynamic games with alternating moves

Game description. There are $N+1$ stages of the game: $t = 0, 1, ..., N$. Sets A and B of possible actions (moves) of players 1 and 2 are given.

• At stage 0 Player 1 makes a move a_0; then Player 2, having observed the Player 1's move a_0, makes a move $b_0 = y_0(a_0)$.

• At stage 1 Player 1 makes a move $a_1 = x_1(a_0, b_0)$ depending on the previous moves a_0 and b_0; then Player 2 makes a move $b_1 = y_1(a_0, b_0, a_1)$, and so on.

• At stage t ($t \leq N$) Player 1 makes a move $a_t = x_t(a^{t-1}, b^{t-1})$ depending on the sequences of his own and rival's previous moves

$$a^{t-1} = (a_0, a_1, a_2, ..., a_{t-1}) \text{ and } b^{t-1} = (b_0, b_1, b_2, ..., b_{t-1})$$

up to time $t-1$, and then Player 2 makes a move $y_t(a^t, b^{t-1})$ depending on the sequences of previous moves

$$a^t = (a_0, a_1, a_2, ..., a_t) \text{ and } b^{t-1} = (b_0, b_1, b_2, ..., b_{t-1}).$$

• The game continues until the stage N, when the players make their last moves a_N, b_N and receive their payoffs $u(h)$, $v(h)$ depending on the whole history of the game

$$h = (a^N, b^N) = (a_0, a_1, a_2, ..., a_N,\ b_0, b_1, b_2, ..., b_N).$$

Strategies. A *strategy of Player 1* is a sequence

$$x_0,\ x_1(a^0, b^0),\ x_2(a^1, b^1),\ x_3(a^2, b^2),\ ...,\ x_N(a^{N-1}, b^{N-1})$$

where x_0 is the initial action of Player 1 and $x_t(a^{t-1}, b^{t-1})$ ($1 \leq t \leq N$) is a function specifying what move

$$a_t = x_t(a^{t-1}, b^{t-1})$$

should be made at stage t given the history (a^{t-1}, b^{t-1}) of the previous moves of the players.

To specify a *strategy of Player 2* one has to specify a sequence of functions

$$y_0(a^0),\ y_1(a^1, b^0),\ y_2(a^2, b^1),\ ..., y_N(a^N, b^{N-1})$$

indicating what move

$$b_t = y_t(a^t, b^{t-1})$$

should be made at stage t given the history (a^t, b^{t-1}) of the previous moves of the players.

We will use the following notation for strategies of players 1 and 2:

$$\sigma = \{x_0, x_1(\cdot), x_2(\cdot), ..., x_N(\cdot)\} \text{ and } \tau = \{y_0(\cdot), y_1(\cdot), ..., y_N(\cdot)\}.$$

Note that for $t = 0$, x_0 is constant and $y_0(a^0)$ depends on $a^0 = a_0$.

Outcome of the game. Given strategies

$$\sigma = \{x_0, x_1(\cdot), x_2(\cdot), ..., x_N(\cdot)\} \text{ and } \tau = \{y_0(\cdot), y_1(\cdot), y_2(\cdot), ..., y_N(\cdot)\}$$

the players apply them recursively, step by step in the alternating order and make their moves

$$a_t = x_t(a^{t-1}, b^{t-1}) \text{ and } b_t = y_t(a^t, b^{t-1})$$

at each stage $t \geq 1$. For $t = 0$, the move of Player 1 is x_0, and the move of Player 2 is $y_0(a^0) = y_0(a_0)$.

The *outcome of the game* $h(\sigma, \tau)$ resulting from the application of the strategies σ and τ is described by the whole history of play

$$h(\sigma, \tau) = (a^N, b^N) = (\underbrace{a_0, a_1, a_2, ..., a_N}_{a^N}, \underbrace{b_0, b_1, b_2, ..., b_N}_{b^N}).$$

Once the outcome $h(\sigma, \tau)$ of the game corresponding to the strategy profile (σ, τ) is known, the players get their payoffs

$$U(\sigma, \tau) = u(h(\sigma, \tau)) \text{ and } V(\sigma, \tau) = v(h(\sigma, \tau)).$$

Chess. Chess is an example of a dynamic game with alternating moves. In this game, possible actions/moves of players 1 and 2 (White and Black) can be identified with positions on the board. When selecting a move, the player selects a new position. The payoffs, depending on the history of play, are defined as follows:

	Player 1 (White)	Player 2 (Black)
White wins	1	0
Black wins	0	1
draw	1/2	1/2

Illegitimate moves (or sequences of moves) lead, by definition, to a zero payoff for the corresponding player.

Remark. Chess is a *finite-stage* game for the following reasons.

There is a finite number of chess-pieces and a finite number of squares on the board, hence there is a finite number of possible positions.

The game automatically terminates as a draw if the same position occurs at least three times, with the same player having to go.

Therefore there exists (possibly very large) N, before which the game ends.

A central question of interest for games under consideration is how to find a winning, or at least unbeatable, strategy for one of the players.

For chess this problem has not been solved and does not seem to be solvable in the nearest future.

Existing supercomputers are not powerful enough for this purpose, and moreover, there are doubts that a sufficiently powerful computing equipment can in principle be developed (in particular, because of the boundedness of the speed of information processing by the speed of light).

The only known facts regarding chess and other finite-stage full-information games with alternating moves are results on the determinacy of such games.

These results say that at least one of the players has a winning (or unbeatable) strategy, but they do not say *which* of the players has such a strategy, and they do not provide an *explicit construction* of such strategies.

Unbeatable and winning strategies. According to the general definition, a strategy σ of Player 1 is called *unbeatable* if

$$U(\sigma, \tau) \geq V(\sigma, \tau) \text{ for any strategy } \tau \text{ of Player 2.}$$

A strategy τ of Player 2 is termed *unbeatable* if

$$V(\sigma, \tau) \geq U(\sigma, \tau) \text{ for any strategy } \sigma \text{ of Player 1.}$$

If the inequalities are strict, then the strategies σ and τ are called *winning.*

The main result regarding the class of dynamic games under consideration is as follows.

Theorem 8.1. *In any finite-stage two-player game with an alternating order of moves of the players, either (i) Player 1 has a winning strategy, or (ii) Player 2 has a winning strategy, or (iii) both have unbeatable strategies.*

We emphasize that in this theorem we speak of *any* finite-stage two-player game with alternating moves, with *any* action sets and *any* payoff functions, so that the formulation of the result has the maximum level of generality.

Application to chess. As a corollary to Theorem 8.1 we immediately obtain the following result.

Theorem 8.2 (Zermelo 1913). *In chess, either White has a winning strategy, or Black has a winning strategy, or both have a strategy guaranteeing at least a draw.*

Strong determinacy. Recall that a game is called *determinate* if at least one of the players has an unbeatable strategy. The game is termed *strongly determinate* if one of the assertions listed in Theorem 8.1 holds. Thus Theorem 8.1 says that any finite-stage two-player game of complete information with alternating moves is strongly determinate.

For a proof of this result see Amir and Evstigneev (2017).

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