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Evolutionary Finance

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Evolutionary Finance

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October 20, 2008

Abstract

Evolutionary finance studies the dynamic interaction of investment strategies in financial markets. This market interaction generates a stochastic wealth dynamics on a heterogenous population of traders through the fluctuation of asset prices and their random payoffs. Asset prices are endogenously determined through short-term market clearing. Investors' portfolio choices are characterized by investment strategies which provide a descriptive model of decision behavior. The mathematical framework of these models is given by random dynamical systems. This chapter surveys the recent progress made by the authors in the theory and applications of evolutionary finance models. An introduction to and the motivation of the modeling approach is followed by a theoretical part which presents results on the market selection (and co-existence) of investment strategies, discusses the relation to the Kelly rule and implications for asset pricing theory, and introduces a continuous-time mathematical finance version. Applications are concerned with simulation studies of the market dynamics, empirical estimation of asset prices and their dynamics, and the evolution of investment strategies using genetic programming.

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1 Introduction

1.1 Motivation and background

Evolutionary finance aims at improving our understanding of the causes and effects of the dynamic nature of financial markets through the application of Darwinian ideas. Market places for risky assets exhibit an unparalleled degree of dynamics and evolution in the behavior and interaction of its participants. The innovations in investment styles, products and the regulatory framework appear to be limitless. All of these changes can be traced back to human endeavor (which tries to achieve intended aims) but, to a similar extent, are they caused by the adaptive, self-organizational and endogenous dynamics of the decisions and interaction of the market participants (which are often unintended consequences). It is this 'life on their own,' which financial markets are often claimed to possess, that our evolutionary approach strives to capture. This chapter surveys the progress made by the authors and their collaborators in this direction of inquiry within the recently established field of evolutionary finance during the last 7 years.

Our approach is rooted in several (quite diverse) lines of research: evolutionary economics, financial economics, economic theory, mathematical finance and dynamical systems theory. The application of evolutionary ideas in the social sciences has a long history. It goes back at least to Malthus, who played an inspirational role for Darwin, see Hodgeson (1993) for a review of this subject. The 1950s saw a renewed interest in this approach with the publications of Alchian (1950), Penrose (1952) and others. This area experienced tremendous developments through the interdisciplinary research conducted in the 1980s and 1990s under the auspices of the Santa Fe Institute which brought together researchers of different backgrounds—economists, mathematicians, physicists and biologists—to study evolutionary dynamics in biology, economics and finance (Arthur et al. (1997), Farmer and Lo (1999), LeBaron et al. (1999), Blume and Easley (1992), Blume and Durlauf (2005)). Their research provided the main source of inspiration and motivation for our work on evolutionary finance.

Evolutionary finance has two defining characteristics: a descriptive approach to the specification of investor and the focus on the dynamics of the wealth distribution. The descriptive modeling of investors shuns any notion of utility and its maximization. The dynamics of investors' wealth is driven by the market interaction of investors and the randomness of asset payoffs. This approach lets actions speak louder than intentions and money speak louder than happiness. Financial practitioners at the cutting-edge of active investment are mainly concerned with beating a benchmark (which is rewarded a bonus) rather than in pursuing some more elusive goals. Evolutionary finance attempts to develop models that reflect this hands-on view to financial markets where the interaction of the investors plays a major role.

Evolutionary modeling overcomes the need to use sophisticated equilibrium concepts; it dispenses with the assumption of a high degree of rationality on the part of the market participants. Both of these assumptions play an important role in classical finance and financial economics despite the fact that they have attracted so much criticism from different quarters over the last century. Evolutionary models of financial markets in contrast rest on a very different view of the behavior of the market par-

ticipants and the interdependence of investment decisions and their performance and, thus, the interaction of the traders. The emphasis in this approach is on a descriptive model of investors to allow for behaviors that are driven by heuristic reasoning and/or behavioral biases, e.g. myopic optimization, dependence of decisions on past performance and other forms of bounded rationality. The choice of the equilibrium concept marks another main shift in the paradigm of how markets work: rather than assuming that all of the investors share the same opinion about the possible future contingencies (and the price of each asset in every possible state), market equilibrium is only invoked in the short term through market clearing at the current date. The advantage of this approach is two-fold: computational and conceptual. Heterogeneity of investors represents the diversity of opinions and types of behavior; short-run goals shift the focus from discounted expected utility to the wealth of investors and its dynamics. A main object of study is the performance of investment styles, in particular within a specific set of strategies. Evolutionary finance opens the door to the study of this line of inquiry without invoking a notion of an equilibrium which requires the agreement of market participants on future price systems.

The investors populating our models can be viewed as heterogeneous agents pursuing particular investment goals. Agent-based models in finance however are typically restricted to a very narrow set of investment strategies. Typically the agent types are defined through myopic mean-variance optimization, the application of technical trading rules such as chartists and fundamentalists; see Chiarella et al. (2009), Hommes and Wagener (2009), Lux (2009). Such an approach has the advantage of explicit demand and supply functions which are derived from standard utility maximization. As demonstrated in this chapter however the class of investment strategies considered might be too restrictive as it ignores better performing investment strategies. It is our aim to maintain the largest degree of freedom in the choice of investment strategies without sacrificing the applicability of random dynamical systems as a modeling framework.

1.2 Applications and real-world implications

Our research aims to contribute to the portfolio choice of investors and to the valuation of financial asset. Both are highly relevant topics for practitioners.

The approach to portfolio choice pursued here is quite different to most of those found in the literature on financial decision making. Rather than offering investment advice for particular tastes of risk, we seek to select investment strategies through the optimality of their asymptotic performance. This performance is dependent on the market interaction of investors which describes the price impact of their strategies. Portfolio choice is therefore informed by objective criteria. The investment recommendation derived in this fashion is closely related to the Kelly rule (the term 'generalized Kelly rule' will be used in this chapter to honor his original contribution). As for any good guide to investment, practitioners have been aware of similar concepts for quite some time—though typically lacking a theoretical foundation. The concept of value investment, which goes back at least 75 years to Graham and Dodd (1934), or stock picking according to relative dividend yield (which even made it as a book title) share the feature that portfolio choice is guided by fundamentals (dividends). The empirical

results of evolutionary portfolio theory presented in this chapter lend some support to the validity of our approach. The ultimate question whether this investment recommendation is normative and should guide individual investors' decisions is left to be decided by those who bet their money—we believe the 30 year debate raged long enough.

A benchmark for asset prices is obtained through the long term outcome of market interaction, the Kelly rule. This finding provides a framework for the valuation of financial assets. The rationale behind this valuation approach is the following: Only if the relative prices correspond to the Kelly benchmark, the asymptotically optimal investment strategy does not achieve excess growth. Otherwise a Kelly investor will reap above average returns. The economic foundation of the evolutionary finance model implies that the benchmark is only meaningful for the tradeable assets. While our approach provides a prediction on the price of one asset relative to that of some other, it does not allow to assess the 'correctness' of the overall valuation of the market. The model therefore provides relative fundamental values which are of particular interest to long-short hedges such as in pairs trading.

1.3 Structure of chapter

The introductory part of this chapter, Section 1.4, discusses the role of dynamics and evolution for evolutionary finance. Section 1.5 explains and demonstrates the basic elements of this approach within Kelly's famous model of horse betting markets. Evolutionary models of financial markets are introduced in detail in Section 2. The theoretical analysis of this class of models is organized in two parts: Section 3 covers models with short-lived assets and Section 4 those with long-lived assets. In both cases the study moves from local dynamics to the (more demanding) global dynamics. Section 4.3 briefly discusses the role of the Kelly rule in dynamic general equilibrium models. A range of applications is presented in Section 5. These comprise simulation studies of the wealth dynamics and the evolution of strategies in combination with genetic programming as well as an empirical study of evolutionary finance and its asset pricing implications. Section 6 highlights recent advances in continuous-time models in evolutionary finance. Section 7 concludes.

Throughout this chapter preference will be given to the heuristic derivation of results. Readers who are interested in the technical details will be provided with references to the relevant articles.

1.4 Dynamics and evolution

Our evolutionary finance approach employs a mathematical framework which is tailored to the description of dynamics in physical and social systems: the theory of random dynamical systems Arnold (1998) (see Schenk-Hoppé (2001) for a survey of applications in economics). The main challenge in the quest for dynamic models of market evolution and trader interaction is the need to break away from the usage of sophisticated equilibrium concepts which are prevalent in economic theory. Standard equilibrium approaches for instance rule out disagreements among agents about future events (e.g. it is common to assume agreement of economic agents about the prices

in each future contingency, Laffont (1989)) and render bankruptcy as the outcome of an agent's deliberate decision. A genuine dynamic and evolutionary model will not remove all surprises the future might hold. These models live from the blunders and unintended consequences of the actions of the individuals populating the model. Survival in an evolutionary struggle is matter of life and death (though, thankfully, traders do not anticipate their demise). The application of evolutionary reasoning requires careful modeling and analysis if one wants to avoid the pitfalls of semantics as forcefully demonstrated by Friedman (1953)'s argument on the price efficiency of markets which was (mistakenly, see De Long et al. (1990)) attributed to the absence of supposedly loss-making irrational traders.

Our aim in advancing evolutionary finance is in particular to impose as few restrictions as possible on the specification of investors and their behavior while, at the same time, accommodating markets with several risky assets. Both these goals shall further be achieved in a truly dynamic model to capture the Darwinian origin of this evolutionary approach. A brief description of the defining characteristics of our evolutionary finance models follows.

Heterogeneity. Diversity in individual's investment behavior is a corner stone of evolutionary finance. The variety of the strategies of market participants (the ecology of the market) makes it possible to analyze the performance of specific investment styles in light of the interdependence of traders through endogenous prices. In the terminology of evolutionary biology, investment types are associated with different species. Two evolutionary forces affect the diversity in the population of investors. On the one hand, variety is reduced by the mechanism of selection. On the other hand, mutation creates novelty in behavior. Since identical behavior in financial markets entails the identical return, it is often possible to choose a representative agent for every investor type. In a finance context it is not import who does what but how much capital is behind a particular investment style. Research aiming at creating descriptive models with heterogeneous agents is, in a sense, perpendicular to the classical financial economics approach in which a single representative agent governs the relations of prices through his indifference.

Strategies. Our approach builds on a model of investment behavior that is purely descriptive. This is at odds with the usual approach in economics in which theories abound to describe the behavior of investors as expressed through their decisions in holding and trading assets and/or consumption: expectations, beliefs, preferences, heuristic decision processes, etc. Investment decisions in evolutionary finance are characterized by investment strategies: budget shares allocated to the wealth invested in the available assets. As long as an investor's total funds are non-zero (e.g. if some collateral is required to borrow) and asset prices are non-zero (e.g. excluding futures), budget shares correspond to portfolio holdings if asset prices are given. In this respect investment strategies are a more primitive concept because they can be defined independent of price systems. Moreover they are easily observable, unlike preferences or behavioral biases. This modeling approach is flexible enough to capture, for instance, agent-based models, general equilibrium models (with and without incompleteness of markets), individual's behavioral biases. Investment strategies are widely used in mathematical finance under the labels 'relative portfolio' (Björk (2004)) or 'trading

strategies' (Pliska (1997)) and they also appear in monetary economics as 'fiscal rules' (Shapley and Shubik (1977)). Their descriptive nature allows for many different interpretations of the behaviors exhibited by the investors. Investment strategies can be constant or, more generally, adapted to some information filtration. They can be governed by a process of selection and mutation for instance using genetic programming (Lensberg and Schenk-Hoppé (2007)).

Dynamic interaction. The performance of strategies is interdependent through their interaction in the market. The action of one investor affects the other investors only through its impact on asset prices. Market clearing is ensured by a pricing rule that gives investors a price impact that is proportional to their wealth. This mechanism implies that the market is shaped to a larger extent by rich rather than poor investors. Two aspects of financial markets are implicitly rather than explicitly in our model: The flow of capital between different investment strategies and the social interaction of investors. Both can be 'accommodated' through the interpretation of the dynamics of investment strategies which leave plenty of scope to address these issues. For instance, the equilibrium of any general stochastic dynamic equilibrium model with incomplete markets can be reproduced by our evolutionary finance model. This merely requires an appropriate specification of the investment strategies. In line with Darwinian ideas we rather prefer to view investors as being 'hardwired' to their strategy while the wealth tied to each investment strategy evolves through market interaction. This perspective highlights the wealth dynamics which acts across investment strategies. The evolutionary finance models discussed in detail in this chapter are like laboratories populated by investment strategies.

Selection and stability. The distribution of wealth across investment strategies exhibits stochastic dynamics. The dynamics of investors' wealth is endogenous because it is driven by random asset payoffs, the trade of assets and consumption goods as well as by the changes that trade entails in portfolio holdings and investments. The wealth dynamics is the most prominent feature of evolutionary finance models. Selection, an elementary Darwinian force, acts through the wealth dynamics in a financial market. Successful investment strategies are those gathering more wealth while strategies losing wealth are rendered unsuccessful by the selection pressure. This interpretation relates to the market selection hypothesis in that the interaction in the market selects strategies through the wealth dynamics. Selection is an asymptotic property of a model, i.e. an outcome that can only be observed in the long-term. Whether selection occurs is a feature related to the stability of dynamical systems (e.g. in a steady state). If a market is characterized by a single strategy (in evolutionary terminology: an incumbent), stability refers to the local dynamics of the wealth distribution when some strategy with little wealth (a mutant) is introduced. The incumbent would constitute a stable market if the mutant is wiped out because the strategy it represents loses all of it's wealth. Instability of a market corresponds to the opposite situation in which the mutant gains wealth.

Evolutionary finance provides a novel approach to *asset pricing*. The stability of markets that are represented by particular investment strategies provides the foundation for an evolutionary asset pricing theory. Suppose there is a model with a unique

investment strategy that is stable against any mutant strategy. Then a market in which assets are priced accordingly exhibits a very strong (evolutionary) stability property. The wealth dynamics provides an actual (rather than fictitious) convergence process for the investors' wealth and therefore for the asset prices. In this sense evolutionary finance can provide an asset pricing theory with sensible stability features. Empirical applications of evolutionary finance are currently at the cutting edge of this research in this field—first results are presented in this chapter.

In contrast to most research related to agent-based modeling of financial markets, the pool of permissible strategies is kept as general as possible. The analytic results will impose different restrictions on the set of investment strategies but simulation studies are, by-and-large, free of these constraints. The combination of the wealth dynamics with type-switching behavior of investors is straightforward. the major advantage to other approaches is that many assets and a richer market ecology can be studied.

In economics the tradeoff between immediate and future consumption in intertemporal models plays a major role in an agent's saving-investment decision while in finance the main focus is the allocation of wealth across investment opportunities. Our evolutionary finance models allow for a strict separation of the consumption and the investment decision through an exogenous (i.e. modeler's) choice of the investors' saving rates. This provides a level playing field for the competition of investors to avoid artifacts such as oversaving. Saving "too much" (i.e. a disproportional amount relative to other investors) due to holding consistently wrong beliefs about future returns is a trait e.g. of general equilibrium models with incomplete markets Blume and Easley (2006). We feel a more narrow view will benefit the study of financial investment. Rather than measuring performance by taking into account consumption amounts, which would fix this problem, our evolutionary finance approach controls consumption through an exogenous and common saving rate. This rate determines the proportion of wealth consumed in each period of time. Every investor spends the same amount per unit of wealth owned which, in turn, entails a level playing field.

1.5 Horse races and the Kelly rule

The main model components and concepts of our evolutionary approach to the study view to financial markets are best introduced and illustrated in a simple model of a betting market. These considerations can be traced back to Kelly (1956) who, among other things, studied optimal investment in parimutuel betting markets in which players repeatedly reinvest their wealth over an infinite time-horizon in win-only bets. The ideas for this line of inquiry on optimal investment were developed by Claude Shannon, the founder of information theory, see Cover (1998).

Consider a race of $K \geq 2$ horses. The odds of the bet 'horse k win' are given by $1 : \alpha_k$, i.e. every \$1 bet on horse k pays $\$\alpha_k$ if this horse wins (and nothing otherwise). The odds correspond to the market's estimate of horse k's chances to win. In a parimutuel betting market without track take one has

$$\frac{1}{\alpha_1} + ... + \frac{1}{\alpha_K} = 1 \tag{1}$$

A risk-free payoff in this betting market is obtained by betting the fraction $1/\alpha_k$ of

one's wealth on horse k, $k = 1, ..., K$. According to (1), the total expenditure is given by $w/\alpha_1 + ... + w/\alpha_K = w$, the bettor's wealth. If, say, horse k wins, the payoff is $\alpha_k \cdot w/\alpha_k = w$ which is equal to the invested fortune.

In a financial market setting, betting corresponds to the holding of assets. The above model can be rephrased as follows. There are $K \geq 2$ assets with prices $p_1, ..., p_K$. Each asset's payoff $A_k(s) \geq 0$ (per unit of the asset) depends on the state of the world $s = 1, ..., S$ which is revealed after all asset purchases are carried out. In a betting market, assets correspond to bets on win and, therefore, $S = K$ and $A_k(s) > 0$ if and only if $s = k$. In other words, these assets are Arrow securities. The odds are given by $p_k : A_k(k)$ or, equivalently, $1 : (A_k(k)/p_k)$; which shows that $\alpha_k = A_k(k)/p_k$. The relation (1) holds if, in each state, the total payoff is equal to the total amount invested. Denoting by q_k the number of asset k held, this condition means that, for all k, $q_k A_k(k) = q_1 p_1 + ... + q_K p_K =: A > 0$. This relation implies (1).

Consider an infinite sequences of horse races in which, for simplicity, the outcome of each race is independent of the previous one. (Horses can have different probabilities of winning a race, though.) Denote the probability of the event that horse k wins by π_k, and let $\pi = (\pi_1, ..., \pi_K)$. The outcome of race t is denoted by s_t, where $s_t \in \{1, ..., K\}$ has probability distribution π for every $t = 1, 2, ...$ Consider a bettor who fixes (once and for all) the share of his wealth to be placed on each particular bet and, moreover, always invests all payoffs received in the previous race. This investment strategy can be formally described by a vector $\lambda = (\lambda_1, ..., \lambda_K)$ with $\lambda_k \geq 0$ and $\sum_{k=1}^{K} \lambda_k = 1$. ($\lambda$ is a vector of portfolio weights.) Starting with initial wealth $w_0 > 0$, the wealth of the bettor after race t is given by

$$w_t = (\alpha_{s_t} \lambda_{s_t}) ... (\alpha_{s_1} \lambda_{s_1}) w_0 \tag{2}$$

The average logarithmic growth rate over t periods is therefore

$$\frac{1}{t} \ln \left(\frac{w_t}{w_0} \right) = \frac{1}{t} \sum_{u=1}^{t} \ln (\alpha_{s_u} \lambda_{s_u}) \tag{3}$$

The strong law of large numbers implies that, as $t \to \infty$, the t-period growth rate (3) converges almost surely to

$$E \ln (\alpha_s \lambda_s) = \sum_{s=1}^{K} \pi_s \ln (\alpha_s \lambda_s) \tag{4}$$

The highest logarithmic growth rate is achieved by the vector of portfolio weights for which $E \ln (\alpha_s \lambda_s)$ is maximal. The Lagrange approach implies that $\lambda_k^* = \pi_k$ for all k.

The vector of portfolio weights $\lambda^* = \pi$ is called the *Kelly rule*. Remarkably, this optimal betting rule does not depend on the odds of the bets. It is clear from (3) that in the case of independent outcomes, the Kelly rule also maximizes the expected value of all *average* logarithmic growth rates. The Kelly investor's wealth will experience a strictly positive growth rate, if the odds do not coincide with the probabilities of paying off, i.e. $1/\alpha_k \neq \pi_k$ for some k. In this case betting with the Kelly portfolio weights yields excess growth because this investor's wealth growth faster than the

average investor (which has growth rate zero). This effect does not occur only if all market's estimates are equal to the objective probabilities, i.e. if $1/\alpha_k = \pi_k$ for all k.

That the expected logarithmic growth rate is a sensible measure of success can be seen as follows. Consider two bettors with portfolio weights λ^1 and λ^2 respectively. Then the wealth of bettor 1 relative to that of bettor 2 evolves as

$$\frac{1}{t}\ln\left(\frac{w_t^1}{w_0^1}/\frac{w_t^2}{w_0^2}\right) = \frac{1}{t}\sum_{u=1}^{t}\ln\left(\frac{\alpha_{s_u}\lambda_{s_u}^1}{\alpha_{s_u}\lambda_{s_u}^2}\right) = \frac{1}{t}\sum_{u=1}^{t}\ln\left(\frac{\lambda_{s_u}^1}{\lambda_{s_u}^2}\right)$$

$$\xrightarrow[t\to\infty]{} E\ln\left(\frac{\lambda_s^1}{\lambda_s^2}\right) =: I_{\lambda^2}(\lambda^1) \tag{5}$$

The term $I_{\lambda^2}(\lambda^1)$ is called the relative entropy of λ^1 with respect to λ^2. If $I_{\lambda^2}(\lambda^1) > 0$, bettor 1's wealth grows exponentially faster than that of bettor 2. In particular $w_t^1/w_t^2 \to \infty$ (almost surely) as $t \to \infty$, i.e. bettor 1 overtakes bettor 2.

Eq. (5) tells the intriguing lesson that the odds (or, equivalently asset prices and payoffs) do not matter for optimal long-term investment. Regardless of the particular odds, the advantage (in terms of the growth rate) of one investor over the other is given by $I_{\lambda^2}(\lambda^1)$. Only objective probabilities and the investors' portfolio rules matter. Moreover, whether one investment strategy is superior to some other can be judged by a pairwise comparison. The total number of active bettors does not play any role.

Interpretation The Kelly rule has several remarkable properties that allow for the interpretation of the result in different contexts.

Equilibrium asset pricing. If the odds of at least one bet do not coincide with the objective probability, there are excess returns (i.e. a strictly positive growth rate) for an investor using the Kelly rule. That is, this investor's wealth will, in the long term, overtake that of any other investor who does not employ the Kelly rule. The equilibrium prices are those that equate the odds and the true probabilities of this event. At these "fair prices" there is no excess return. Every investor who employs the Kelly rule has a growth rate of wealth equal to zero, and any other investor experiences a negative growth rate.

Market selection. The wealth dynamics of investors provides a mechanism for the comparison of their performance in the market. An investor with a higher growth rate than a competitor is selected by the market in the sense that the relative wealth of investors with lower growth rates tend to zero. The market selects for investors employing the Kelly rule. The analysis is greatly facilitated here because the interaction of investors through prices does not play any role: performance can be quantified solely by using the objective probabilities (see the relation (5)).

Betting your beliefs. The best decision of an investor, who strives to maximize his growth rate but is not informed about the true probabilities, is to choose a portfolio rule according to his estimate (or partial knowledge) of the true probabilities, i.e. to "bet his beliefs." An investor with a better estimate or knowledge of the vector π (in terms of the entropy) than other investors will achieve a higher growth rate and, thus, overtake the investors with inferior estimates. In the above model, Bayesian updating presents the optimal way of learning about the true probabilities.

Log-optimum investment. The fact that the Kelly rule maximizes the expected logarithmic growth rate as well as *any* expect average logarithmic growth rate, can be used to characterize the Kelly rule as the one that maximizes, in any point in time, the logarithmic growth rate. We will later see that this trait is specific to win-only betting markets (i.e. a financial market consisting only of Arrow securities). A general mathematical theory on log-optimum investment has been developed by Breiman (1961), Thorp (1971), Algoet and Cover (1988) and Hakansson and Ziemba (1995).

Generalizations Consumption (as a share of wealth) is easily accommodated. Suppose bettor i reinvests the constant fraction $0 < \delta^i \leq 1$ of his wealth in every one race. Then considerations completely analogous to the above show that bettor 1 overtakes bettor 2 if and only if $I_{\delta^2\lambda^2}(\delta^1\lambda^1) = E\ln\left(\delta^1\lambda_s^1/\delta^2\lambda_s^2\right) > 0$. Even if λ^1 is closer to λ^* than λ^2, a too small δ^1 (relative to δ^2) can ensure that bettor 2 overtakes bettor 1. It is clear that more economic content can be added to the specification of investors in the model, e.g. econometric learning models. The simple link between the absolute performance of an investment strategy and the consumption rate as well the closeness to the Kelly rule enables a study of the market selection hypothesis in this framework. A detailed coverage is given by Blume and Easley (2009).

Kelly's contribution to portfolio choice has stirred an amazing controversy within financial economics. The main adversary in this debate is Samuelson (1979) who questioned the value of the Kelly rule as an investment advice on the ground that "we should not make mean log of wealth big though years to act are long." In essence the critique is that you should maximize your utility function rather than to base your investment decision on some other criterion. This is certainly correct, but fails to appreciate that Kelly's results are not necessarily normative but rather descriptive. This is in particular true if the issue of selection of investment strategies (in connection with the market selection hypothesis) is discussed. In this view, the Kelly rule (as well as the growth rates) provides a benchmark which is available to an outside observer.

2 Evolutionary models of financial markets

There are two main classes of models, both with proven potential. This section provides an outline of these models by introducing its principal components, followed by a thorough discussion of the underlying assumptions. The section concludes with an outline of the wealth and price dynamics.

The fundamental difference between these two model classes is the life span of the assets: assets either live for one period (short-lived assets) or infinitely many periods (long-lived assets). Short-lived assets are entitlements to a random payoff. They are issued at some point in time, pay out at the beginning of the subsequent period and then become worthless (i.e. disappear and are issued again). Examples are bets at the horse track or options with a one period maturity. A detailed discussion of this model with Arrow-type securities is provided in Section 1.5. Long-lived assets produce a random payoff stream from the day of issue on which lasts until eternity. Since these assets do not expire or disappear, their (future) value is positive and they are traded among the investors. The classical example is that of dividend-bearing stocks. In the

first case an investor's income is only from asset payoffs, while for long-lived assets investors receive dividend income as well as capital gains (or losses) from price changes in the assets.

2.1 Components of the models

Both types of evolutionary finance models (short- and long-lived assets) use the same components which are explained in detail here.

Time All models discussed here are placed within a discrete-time framework. Time is indexed with $t = 0, 1, 2, ...$, with $t = 0$ being the initial time period.

Randomness The randomness of asset payoffs is modeled through a sequence of random variables s_t, $t = 0, 1, ...$, with a finite state space S. $s_t = 1, ..., S$ describes the "state of the world" at time t. It is convenient (and without loss of generality) to assume that there is an infinite past as well, i.e. states of the world s_t are also defined for $t = -1, -2, ...$ The state is either an i.i.d. (independent and identically distributed) process with distribution $\pi(s) = P\{s_t = s\} > 0$ or, more generally, a time-homogenous Markov process with transition probabilities $\pi(s|\hat{s}) = P\{s_{t+1} = s|s_t = \hat{s}\} \geq 0$. The state s_t should be seen as a proxy of a rather complex set of variables characterizing investors' information. At each point in time t, the vector $s^t = (..., s_{t-1}, s_t)$ denotes the history of events.

Assets There are $K \geq 1$ assets, each in unit supply. Asset k's payoff at time t is given by $A_k(s_t)$. The asset payoff is in terms of a (perishable) consumption good—just as in Lucas (1978). This assumption in particular ensures that the assets are the only store of value. The dependence on s_t is responsible for the randomness of the payoff. Throughout the remainder of this chapter we will assume that

$$A_k(s) \geq 0 \quad \text{and} \quad \sum_{k=1}^{K} A_k(s) > 0 \tag{6}$$

for all k and all s. It is further convenient (and customary) to assume the absence of redundant assets. This condition ensures that different portfolios have different payoff streams and, thus, a unique relationship. The functions $A_1(\cdot), ..., A_K(\cdot)$ restricted to the set $\{s \in S : \pi(s) > 0\}$ are linearly independent. Since $\pi(s) > 0$ for all $s = 1, ..., S$, there are no redundant assets if and only if the matrix $(A_1(\cdot), ..., A_K(\cdot))$ has full rank.

Assets are called short-lived, if they pay off only once and then become worthless. They are called long-lived, if they produce a payoff stream that, in each period in time, has a strictly positive probability of being strictly positive.

Strategies/Investors There are $I \geq 1$ investors who can trade in the K assets at every point in time t. Investor i's wealth at time t is denoted by w_t^i, the initial endowment being $w_0^i \geq 0$. An investor's wealth can change because of (a) receipts of asset payoffs, (b) changes in asset prices and (c) expenditures for consumption. Each investor is characterized by an investment strategy, a time- and history-dependent vector of portfolio weights. Investor i's investment strategy is denoted by

$$\lambda_t^i = (\lambda_{1,t}^i, ..., \lambda_{K,t}^i),\ \lambda_t^i = \lambda_t^i(s^t), \quad t \geq 0, \tag{7}$$

with

$$\lambda^i_{k,t} > 0 \quad \text{and} \quad \sum_{k=1}^{K} \lambda^i_{k,t} = 1 \tag{8}$$

The value of $\lambda^i_{k,t}$ is investor i's budget share allocated to the investment in asset k (obtained either through purchases or reduction of a position). Non-negativity of the budget shares means that short-selling is not permitted.

It will be assumed throughout the following that the "pool" of the I strategies only contains strategies which are different from each other. As usual in evolutionary theory, the focus is on parts of a population pursuing a particular type of behavior rather than on the individual. In a finance context this identification is straightforward. All individuals who follow the same investment strategy are considered as owners of an investment fund pursuing that strategy. Each individual's wealth is equal to a fraction (the share of his initial contribution) of the fund's current wealth.

Budget The budget of investor i available for the purchase of assets at time t is denoted by b^i_t. This budget depends on the investor's income and consumption. If investor i has a saving rate $0 \leq \rho^i \leq 1$, his budget is $b^i_t = \rho^i w^i_t$. The expenditure on consumption is $(1-\rho^i)w^i_t$. It will be assumed that there is a common (constant) saving rate ρ for all the investors. The endowment is in wealth.

Prices Asset prices $p_{k,t}$ at any point in time t are determined by market clearing. Given every investor's portfolio weights λ^i_t and the vector $b_t = (b^1_t, ..., b^I_t)$ of the budget of investors which is available for investment, the price of asset k is

$$p_{k,t} = \langle \lambda_{k,t}, b_t \rangle := \sum_{i=1}^{I} \lambda^i_{k,t} b^i_t \tag{9}$$

where $\lambda_{k,t} = (\lambda^1_{k,t}, ..., \lambda^I_{k,t})$. Given a common saving rate ρ, the price of asset k at time t is given by $p_{k,t} = \rho \langle \lambda_{k,t}, w_t \rangle$.

Portfolios After transaction at prices $p_{k,t} > 0$, investor i's portfolio is given by

$$\theta^i_{k,t} = \frac{\lambda^i_{k,t} b^i_t}{\langle \lambda_{k,t}, b_t \rangle} \tag{10}$$

i.e. $\theta^i_{k,t}$ is equal to the budget of investor i for the purchase of asset k divided by the price of asset k. Aggregating (10) over investors one can verify that the total demand is equal to the total supply: $\sum_i \theta^i_{k,t} = 1$. With a common saving rate ρ, $\theta^i_{k,t} = \rho \lambda^i_{k,t} w^i_t / \langle \lambda_{k,t}, \rho w_t \rangle = \lambda^i_{k,t} w^i_t / \langle \lambda_{k,t}, w_t \rangle$.

2.2 Discussion of the assumptions

A few comments relating these definitions to the literature are in order.

Investment strategies are specified as non-negative budget shares. This precludes short selling of assets. The assumption is necessary to rule out bankruptcy as well as undefined asset prices. In particular bankruptcy (i.e. negative net worth) would

be prevalent in a dynamic model in which perfect foresight is absent, see also De Giorgi (2008). The absence of demand functions further prevents the usual mechanism that yields strictly positive asset prices. This assumption can therefore be seen as a necessary limitation when considering a behavioral model in a dynamical systems setting.

Asset prices are determined with a market-clearing mechanism which, surprisingly perhaps, does not require demand functions. Remarkably, this pricing rule simultaneously clears any number of markets. This is in stark contrast to general equilibrium models and even to most agent-based models. An economic interpretation of this market clearing approach is that of fiscal rules as introduced by Shapley and Shubik (1977). In financial mathematics, the relation (9) between prices and strategies is a consequence of the self-financing constraint on portfolios.[1] Prices are linear combinations of the investors' strategies with weights determined by the investors' wealths. The prices will therefore resemble the rich investors' strategies rather than those who are poor. In the extreme case in which all investors but one have no wealth, the prices will be determined by the single investor with capital. The price rule (9) governs the market interaction of investors. Each investor has an impact on the price proportional to his wealth.

Trade between agents takes place as an exchange of assets and the consumption good. An investor will therefore become richer, if he has above-average dividend income and capital gains or if the superior performance in one source of income outweighs inferiority in the other.

The absence of a market-clearing mechanism for the consumption good is explained by Walras' law. All asset markets clear, investors exhaust their budgets and, thus, the remaining market for the consumption good also clears. It will be convenient to use the price of the consumption good as the numeraire (and thus set it equal to one).

Asset payoffs are made in a perishable consumption good, an assumption which is common in financial economics, Lucas (1978). Its main advantages in the present context are that only the assets can be used for the intertemporal transfer of wealth and that there is no growing stock of money which could inflate prices. In agent-based models with one stock and money, an increase in the money supply does not affect the return on the stock because of agents' CARA utility functions. This specification of preference ensures the independence of investors' appetite for risk from the level of wealth, see e.g. Hommes and Wagener (2009).

The careful treatment of dividends as consumption good is inspired by economics. The is a clear preference for closed models in the sense that every good is accounted for (and equations balance). Assets can be interpreted as firms which are endowed with an initial capital stock that is worked to produce goods. Here the produce is a generic consumption good. Each asset could be viewed as a sector of the economy, with the aggregate payoff being the economy's gross domestic product. Our analysis will focus on the case in which the relative payoffs possess some degree of stationarity.

Whether the assets are short- or long-lived will have a substantial impact for the wealth dynamics in evolutionary finance models. Since the agents are boundedly ra-

[1] See e.g. Section 6.2 in Björk (2004), and Sections 2.5 and 5.6 in Pliska (1997) as well as the discussion in Section 6 in this chapter.

tional, capital and dividend gains play different roles. The presence of capital gains (or losses) strengthens the link between the market dynamics and the individual investor's performance. Both models will therefore display different dynamics. In general equilibrium models in which economic agents have perfect foresight (Laffont (1989)) however these two cases essentially coincide. Payoffs $A_{k,t+1}(s^{t+1})$ and prices $p_{k,t+1}(s^{t+1})$ can be replaced, in equilibrium, by cum-dividend prices (long-lived assets) or a cum-price dividends (short-lived assets): $p_{k,t+1}(s^{t+1}) + A_{k,t+1}(s^{t+1})$. Then the same allocation can be obtained after appropriate change of the agents' portfolios. What matters for the equilibrium dynamics is the span of the dividend matrix.

2.3 Outline of the dynamics

A brief exposition is given of the dynamics in the two models (short- resp. long-lived assets). The purpose of this section is to provide some intuition for this modeling approach without going into technical detail—this is reserved for later.

If assets are short-lived, the investment income only consists of dividends. The wealth dynamics of investor i can be written as

$$w^i_{t+1} = \sum_{k=1}^{K} A_{k,t+1}(s_{t+1})\, \theta^i_{k,t} \tag{11}$$

For long-lived assets, changes in the asset prices will affect the investors' wealth in addition through capital gains and losses. One has the dynamics

$$w^i_{t+1} = \sum_{k=1}^{K} \left(A_{k,t+1}(s_{t+1}) + p_{k,t+1}\right) \theta^i_{k,t} \tag{12}$$

Clearly, if assets do not have a resale value, i.e. $p_{k,t+1} = 0$, then (12) is identical to (11).

The market interaction of all the investors is via their impact on the asset prices. According to (9), $p_{k,t} = \langle \lambda_{k,t}, b_t \rangle$ where b^i_t denotes investor i' budget. If there is a common saving rate ρ, one has $p_{k,t} = \rho\langle \lambda_{k,t}, w_t \rangle$. The price of each asset therefore depends on the wealth distribution $w_t = (w^1_t, ..., w^I_t)$. In fact the price represents a wealth-weighted strategy. When adjusting the portfolio, the number of shares held (relative to wealth) by an investor will depend linearly on his strategy. If the budget share of the strategy exceeds the price, the investor will have a higher exposure to that asset than to those which are assigned a smaller share.

Let us assume for the time being that both dynamics (11) and (12) are well-defined. (Details are left for later.) Then, for a given set of strategies, a wealth distribution $w_t = (w^1_t, ..., w^I_t)$ is mapped into a new distribution of wealth across the investors $w_{t+1} = (w^1_{t+1}, ..., w^I_{t+1})$ simply by drawing a state of nature s_{t+1} and applying (11) resp. (12). The evolution of the wealth distribution is defined by a dynamical system with a random component (the state of nature) outside the control of the economic agents. In mathematical terms, each of these equations defines a random dynamical system, Arnold (1998).

Selection, survival and stability will all be defined in terms of this wealth dynamics. Evolutionary stability will, in addition, allow for the enlargement of the number of investors. The concept of incumbents and mutants is embedded for instance as follows. With two investors with wealth (w_t^1, w_t^2), the incumbent-mutant situation corresponds to the case in which w_t^1/w_t^2 is either very large (investor 1 being the incumbent) or very small (investor 1 being the mutant). Stability refers to the convergence of the wealth distribution after a small perturbation of the wealth distribution that moves it away from a steady state.

3 An evolutionary model with short-lived assets

This section introduces an evolutionary finance model with short-lived assets. It is a direct generalization of the Kelly model of a parimutuel betting market discussed in Section 1.5. The presentation draws on Amir et al. (2005), Evstigneev et al. (2002), and Hens and Schenk-Hoppé (2005b).

The main innovation is the introduction of incomplete markets in this framework. In contrast to the above, it turns out that the growth rate of an investor's wealth depends on the asset prices. Since prices matter in this setting, the issue of the market interaction (and the price dynamics it entails) becomes important. The distinctive property of short-lived assets is the absence of a resale value. Each asset pays off one period after its issue and then become worthless. This requires, at every period in time t, the (re-)issue of new assets. These assets are just like a lottery tickets or, as discussed in detail above, bets on win. After the winners received their payoff the tickets are worthless.

The approach is incremental, with the most simple version of the evolutionary model with short-lived assets presented first. This basic setting already provides a good intuition for evolutionary finance models without burdening the reader with too much notation and technicalities. Section 3.5 discusses a much more general case with adapted investment strategies and the state of the word following a Markov process. The assumptions introduced in Section 2.1 are supposed to hold.

3.1 The model

Suppose the state of the world follows an i.i.d. process. Asset payoffs depend only on the current state of nature, i.e. one unit of asset k bought at time t pays out $A_k(s_{t+1})$ with s_{t+1} being the state of nature that is revealed after all trade is completed in period t. Investors employ constant proportions strategies, i.e. portfolio weights are fixed once and for all, $\lambda^i = (\lambda_1^i, ..., \lambda_K^i)$, $i = 1, ..., I$. There is no consumption but investors reinvest all receipts in any one period.

The wealth in period $t+1$ of investor i is determined by the portfolio purchased in period t and the realization of the random asset payoffs. The relation between the portfolio $\theta_{k,t}^i$ and the wealth w_{t+1}^i is given by

$$w_{t+1}^i = \sum_{k=1}^{K} A_k(s_{t+1})\, \theta_{k,t}^i \tag{13}$$

Inserting the definition of the portfolio (10), one obtains the random dynamics of the wealth of investor i as

$$w_{t+1}^i = \sum_{k=1}^{K} A_k(s_{t+1}) \frac{\lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle} \tag{14}$$

These dynamics exhibit the market interaction of investors. The right-hand side of (14) depends on the distribution of wealth $w_t = (w_t^1, ..., w_t^I)$ across investors as well as every investor's strategy. The evolution of investors' wealth is therefore interdependent, with the dependence being caused by each investor's impact on the asset prices.

The particular circumstances under which the analysis for the Kelly rule applies is apparent from (14). Testing for overtaking by calculating w_{t+1}^i / w_{t+1}^j yields the expression

$$\frac{w_{t+1}^i}{w_{t+1}^j} = \frac{\sum_{k=1}^{K} A_k(s_{t+1}) \lambda_k^i / p_{k,t}}{\sum_{k=1}^{K} A_k(s_{t+1}) \lambda_k^j / p_{k,t}} \frac{w_t^i}{w_t^j} \tag{15}$$

The asset prices only cancel if, in every state of the world, exactly one asset has a strictly positive payoff (and all the others have zero). In other words, if the market consists only of Arrow securities which, in particular, implies completeness of the market. In this sense, in incomplete markets prices matter (for relative growth and, thus, for survival). The study of the long-term dynamics will be more involved. Let us first transform the problem into one which is more convenient to analyze.

An investor with initial wealth $w_0^i > 0$ has, by our assumption on strictly positive budget shares (8), strictly positive wealth at every point in time, i.e. $w_t^i > 0$ for all $t \geq 0$, see (14). Further, since the aggregate supply of each asset is equal to one, the aggregate or total wealth of investors is

$$W_{t+1} = \sum_{i=1}^{I} w_{t+1}^i = \sum_{k=1}^{K} A_k(s_{t+1}) \tag{16}$$

To this end we obtain the dynamics of the relative wealth

$$r_t^i = w_t^i / W_t$$

as

$$r_{t+1}^i = \sum_{k=1}^{K} R_k(s_{t+1}) \frac{\lambda_k^i r_t^i}{\langle \lambda_k, r_t \rangle} \tag{17}$$

where

$$R_k(s) = \frac{A_k(s)}{\sum_{n=1}^{K} A_n(s)}$$

The random functions $R_k(s)$, the *relative asset payoff*, inherit all properties from the original payoff process $A_k(s)$ and, thus, satisfies our assumptions. In addition one has $\sum_k R_k(s) = 1$ for all $s = 1, ..., S$. In matrix notation,

$$R = \begin{pmatrix} R_1(1) & \dots & R_K(1) \\ \vdots & \dots & \vdots \\ R_1(S) & \dots & R_K(S) \end{pmatrix} \tag{18}$$

The dynamics (17) lives on the simplex

$$\Delta^I = \left\{ x = (x^1, ..., x^I) \in \mathbb{R}^I : \; x^i \geq 0, \; \sum_{i=1}^{I} x^i = 1 \right\}$$

The initial state $r_0 = (r_0^1, ..., r_0^I) \in \Delta_+^I$ is given by $r_0^i = w_0^i / W_0$. The vector of wealth shares $r_t = (r_t^1, ..., r_t^I)$ at time t depends on the entire history of states of the world, i.e. $r_t = r_t(s^t)$, see (14). As explained above, $r_0 \in \Delta_+^I$ implies $r_t \in \Delta_+^I$ for all t and s^t. Indeed, (14) defines a random dynamical system (Arnold (1998)) on a simplex. Illustrations are provided in Figures 1 and 2

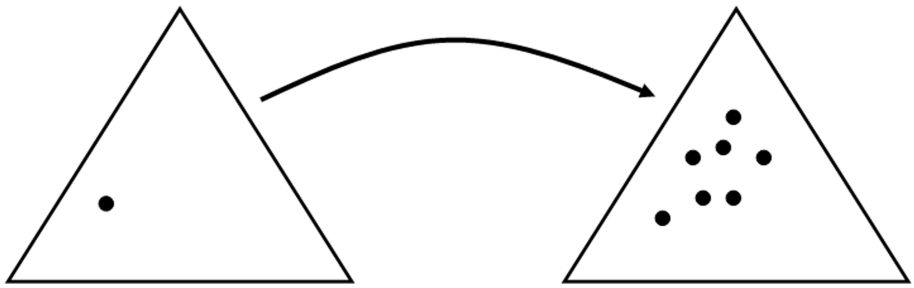


Figure 1: Graphical representation of the map defined in (17) for $I = 3$. The state r_{t+1} depends on the realization of the state of the world s_{t+1}. The simplex on the right shows all possible future states r_{t+1} for a given vector of wealth shares r_t; the actual state r_{t+1} observed depends on the realization of the random event s_{t+1}.

Each vertex of the simplex corresponds to a state in which all but one component of the vector of wealth shares is equal zero. An investment strategy λ^i is therefore associated with the corresponding vertex e^i of the simplex. It is straightforward from (17) that every vertex is a *fixed point* because $r_0^i = 0$ implies $r_t^i = 0$ for all t. In the steady state e^i only investor i has positive wealth and the asset prices coincide with his portfolio shares. The fixed points of these dynamics are of central interest because they correspond to particular investment strategies. The absence of redundant assets ensures that there are no fixed points in the interior of the simplex, i.e. the dynamics cannot 'get stuck' (Hens and Schenk-Hoppé, 2005b, Proposition 1). The same considerations show that every face of the simplex Δ^I is invariant under the dynamics (17). A face of the simplex corresponds to a situation in which certain investors have no wealth and do not impact prices. There is however a (non-trivial) wealth dynamics among the remaining investors.

Selection The criterion of overtaking, where investor i's wealth grows faster than that of investor j ($w_t^i / w_t^j \to \infty$ as $t \to \infty$), translates into the convergence of the vector of wealth shares r_t towards a face of the simplex or a vertex. For instance, if

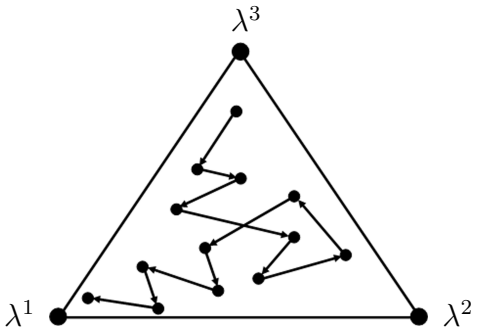


Figure 2: Random dynamics of the relative wealth r_t in (17) with $I = 3$. The vertices correspond to steady states in which the respective investment strategy's wealth share is equal to 1.

$w_t^1/w_t^j \to \infty$ for every $j \neq 1$ then $r_t \to (1, 0, ..., 0) = e^1$. As a formal definition, one says investment strategy λ^i (represented by investor i) is *selected*, if $\lim_{t\to\infty} r_t \to e^i$ almost surely, where e^i is the ith vertex (i.e. the vector with all components equal to zero except for the ith component which is one). The qualifier "almost surely" will mostly be dropped in the following.

Selection and the stability of fixed points are closely linked. In essence a fixed point is *stable* if the wealth shares converge back to the steady state after a small perturbation. This small displacement of the fixed point is used as the initial state. Selection will often happen exponentially fast. In this case stability can be detected through linearization at the fixed point.

Our analysis of the model considered in Kelly (1956), Section 1.5, revealed that a *pairwise* comparison of investors suffices to analyze the issue of selection. This might not be appropriate in the general case in which the pool of investors matters for the wealth dynamics of each market participant. A notion of stability of strategies (i.e. the vertex that it represents) in markets with a different numbers of investors being present is required. Such a market will be referred to as a "pool of strategies."

Definition of stability An investment strategy λ^i is called:

• *globally stable* in a given pool of strategies, if the fixed point e^i is globally stable: for every $r_0 \in \Delta$ with $r_0^i > 0$, $\lim_{t\to\infty} r_t = e^i$;

• *(locally) stable* in a given pool of strategies, if the fixed point e^i is (locally) stable: there exists a (random) neighborhood of e^i such that $\lim_{t\to\infty} r_t = e^i$ for each initial r_0 in this neighborhood;

• *globally evolutionary stable*, if λ^i is globally stable in any pool of investment strategies. In line with our assumptions, all investment strategies in this pool have to be different to λ^i. (Local) evolutionary stability is defined analogously.

All of these notions relate to the idea of mutant strategies entering the market. Local concepts correspond to mutants possessing little wealth initially, while global refers to a perturbation of the wealth distribution which is not necessarily small. Market selection, which is often referred to in contexts similar to the one considered here,

can be interpreted as both local or global property of the wealth dynamics. The most demanding requirement is the globally evolutionary stability of an investment strategy.

3.2 Analysis of local dynamics

A mathematical analysis of the local stability properties of investment strategies requires advanced methods from random dynamical systems theory, see Hens and Schenk-Hoppé (2005b). The local stability of a fixed point can be derived, under certain assumptions, from the linearization of a random dynamical system at this point (analogous to deterministic dynamical systems). The linearization allows to infer (local) logarithmic growth rates, called Lyapunov exponents (or eigenvalues for deterministic systems), of the original system. If all Lyapunov exponents of the linearized system are strictly negative, the local dynamics drives the state back to the fixed point. But if at least one Lyapunov exponent is strictly positive, the dynamics does not provide this pull—the fixed point is unstable.

Fortunately a heuristic derivation of the local stability analysis is available. It is presented in the following. To derive a criterion for the local stability of a constant investment strategy λ^i, suppose that r_t is close to e^i. Then the (relative) price of asset k is given by

$$q_{k,t} = \langle \lambda_k, r_t \rangle = \sum_{j=1}^{I} \lambda_k^j r_t^j \approx \lambda_k^i$$

Inserting this approximation in (17), one finds

$$r_{t+1}^j \approx \sum_{k=1}^{K} R_k(s_{t+1}) \frac{\lambda_k^j r_t^j}{\lambda_k^i} = \left(\sum_{k=1}^{K} R_k(s_{t+1}) \frac{\lambda_k^j}{\lambda_k^i} \right) r_t^j \tag{19}$$

for every $j = 1, ..., I$. Arranging these approximations in the form of a linear equation (with vector r_t) gives the variational equation, which is stochastic. The logarithmic growth rate of investor j's wealth share is therefore approximated by

$$\frac{1}{t} \ln \left(r_t^j / r_0^j \right) = \frac{1}{t} \sum_{u=0}^{t-1} \ln \left(\frac{r_{u+1}^j}{r_u^j} \right) \approx \frac{1}{t} \sum_{u=0}^{t-1} \ln \left(\sum_{k=1}^{K} R_k(s_{u+1}) \frac{\lambda_k^j}{\lambda_k^i} \right)$$

$$\underset{t \to \infty}{\longrightarrow} E \ln \left(\sum_{k=1}^{K} R_k(s) \frac{\lambda_k^j}{\lambda_k^i} \right) =: g_{\lambda^i}(\lambda^j) \tag{20}$$

The growth rate has a straightforward interpretation. Any mutant competes in a market in which the prices are determined by the incumbent's strategy. From the perspective of a potential entrant to the market, he will act in a market in which the dividend yields (which correspond to the asset returns here) are given and not influence by his actions. The application of this finding to the evolutionary stability of strategies is detailed below.

The analysis also led us full circle back to the pairwise comparison of investment strategies. This finding is surprising because, as already pointed out above, a direct

attack using the overtaking criterion fails to work. Indeed the above considerations show that locally the impact of mutants on the price is negligible (a second order effect, in economic terms). Close to a steady state (which is where selection happens, if it does) a one-to-one comparison of strategies suffices to gauge the dynamics of a multi-investor setting. In mathematical terms, the dynamics is locally decoupled, i.e. the growth rate $g_{\lambda^i}(\lambda^j)$ depends only on the strategies of investors i and j. This is a consequence of the fact that the matrix appearing in the variational equation (the dynamics of the linearization) is diagonal.

The information contained in (20) is easy to extract. If the growth rate is strictly negative, investor j's wealth share declines and eventually goes to zero. On the other hand, if (20) is strictly positive for some j, investor j's wealth share increases. From the perspective of investor i, if there is one $j \neq i$ such that (20) is strictly positive, then investor i's wealth share decreases, i.e. it does not converge to one. A potential shortcoming of this approach is that it only measures speed at an exponential scale. Slower convergence/divergence speeds will not be detected. For our purpose, however, this plays no role.

The following proposition summarizes our discussion, see (Hens and Schenk-Hoppé, 2005b, Proposition 2).

Theorem 3.1. *Consider the growth rate*

$$g_{\lambda^i}(\lambda^j) = E \ln \left(\sum_{k=1}^{K} R_k(s) \, \frac{\lambda_k^j}{\lambda_k^i} \right) \tag{21}$$

The investment strategy λ^i is

(i) stable, if $g_{\lambda^i}(\lambda^j) < 0$ for all $j \neq i$;

(ii) unstable, if $g_{\lambda^i}(\lambda^j) > 0$ for some $j \neq i$.

A criterion for the local evolutionary stability of an investment strategy is a direct application of Theorem 3.1. The investment strategy λ is locally evolutionary stable if

$$E \ln \left(\sum_{k=1}^{K} R_k(s) \, \frac{\mu_k}{\lambda_k} \right) < 0 \text{ for all } \mu \neq \lambda \tag{22}$$

where μ is an investment strategy satisfying the assumptions in Section 2.1.

Stability of markets The above result provides a simple criterion to test for the stability (or instability) of a market which is characterized by particular asset prices. Using that any price system can be represented by a situation in which just one investment strategy owns all of the wealth, the stability properties of the corresponding fixed point reflect that of the system of asset prices.

Co-existence of strategies The co-existence of strategies corresponds to a situation with the feature that, in a given pool of strategies, *all* of the investment strategies are locally unstable. Then selection fails to hold and no strategy can wipe out (or be wiped out by) its competitors. As Theorem 3.1 asserts, co-existence of investment strategies is linked to the growth rates in the neighborhood of steady states. A negative growth rate of investor i close to the steady state e^i means that the prices turn against the

richest investor's strategy. He cannot grow at the prices induced by his investment strategy.

The particular role played by the price mechanism in the interaction of investors is made explicit by Theorem 3.1. Whether an investor (a mutant) can increase his wealth at the expense of the incumbent depends how well his strategy performs at the prices that are induced by the incumbent's strategy. Interestingly perhaps, the concept of evolutionary spite does not have any bite here because the total payoff is independent of the decisions of the investors.

3.3 An example

An illustration is provided of the selection and co-existence of strategies. This simple example also highlights the feature that prices 'can turn against you.'

Let the payoff matrix be given by

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 3 \end{pmatrix} \quad \Rightarrow \quad R = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$$

The market in incomplete with two assets and three states. States of the world are i.i.d. with $\pi(s) = 1/3$ for $s = 1, 2, 3$. Consider two scenarios with two resp. three investment strategies. The strategies are constant and given by

$$\lambda^1 = (1/2, 1/2),\ \lambda^2 = (1/4, 3/4),\ \lambda^3 = (1/3, 2/3) \tag{23}$$

In scenario 1, λ^1 and λ^2 are present in the market; both endowed with equal initial wealth shares. In scenario 2, the strategy λ^3 is added to this set of investment strategies; the initial wealth share of the new strategy is 10% while the two others equally share the remainder. A typical simulation run is depicted in Figure 3. The left panels show all strategies wealth shares and the right panels the relative prices of both assets for each case.

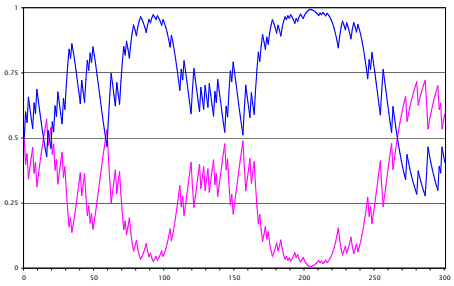
Coexistence of investment strategies occurs in scenario 1 with just two strategies. The addition of the investment strategy λ^3 leads to a very different outcome: λ^3 is selected because the wealth of the two other strategies tends to zero.

The growth rates of investment strategies in scenario 1 can be equated as

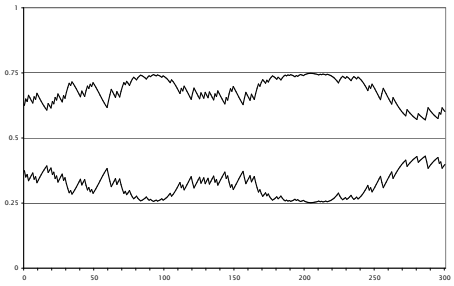
$$g_{\lambda^1}(\lambda^2) = \frac{1}{3}\left[2\ln\left(\frac{1}{2}\,\frac{1/4}{1/2} + \frac{1}{2}\,\frac{3/4}{1/2}\right) + \ln\left(0\,\frac{1/4}{1/2} + 1\,\frac{3/4}{1/2}\right)\right] \approx 0.13515 > 0$$

$$g_{\lambda^2}(\lambda^1) = \frac{1}{3}\left[2\ln\left(\frac{1}{2}\,\frac{1/2}{1/4} + \frac{1}{2}\,\frac{1/2}{3/4}\right) + \ln\left(0\,\frac{1/2}{1/4} + 1\,\frac{1/2}{3/4}\right)\right] \approx 0.056633 > 0$$

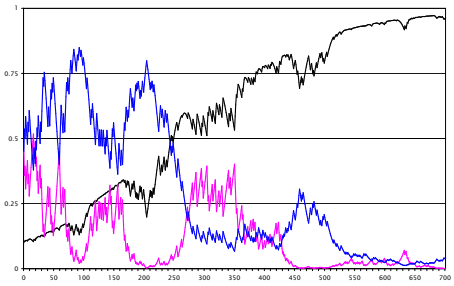
Both investment strategies λ^1 and λ^2 are locally unstable; selection cannot work and these two strategies coexist as illustrated in Figure 3 (a). The underlying cause for these dynamics can be traced to the price dynamics. If the investment strategy λ^1 owns almost all wealth, the price of asset 2 becomes too low, which is to the advantage of strategy λ^2 that places more wealth on asset 2. In the opposite situation in which λ^2 owns almost all of the wealth, asset 1 becomes too cheap. The investment strategy λ^1



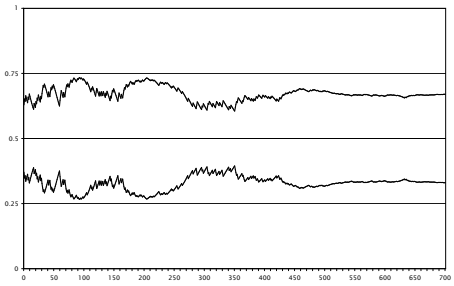
(a) Scenario 1: wealth share dynamics



(b) Scenario 1: asset price dynamics



(c) Scenario 2: wealth share dynamics



(d) Scenario 2: asset price dynamics

Figure 3: Dynamics of the evolutionary finance model with short-lived assets defined in Section 3.3. Strategies are defined in (23). Scenario 1: two strategies (λ^1 and λ^2), time periods 0-300. Scenario 2: three strategies (λ^1, λ^2 and λ^3), time periods 0-700. Both simulation runs use the same time series of states s_t, $t = 0, ..., 700$.

benefits from this price system because it puts a higher share on that asset. This line of reasoning is confirmed by the price dynamics. In the time period 150-225, during which strategy λ^1 is relatively poor, the price for asset 1 is lower than at any other point in time. One also observes consistent price fluctuations in Figure 3 (b). These findings highlight the dynamic interaction of the investment strategies; an interaction solely through the price system.

In the second scenario with the additional investment strategy $\lambda^3 = (1/3, 2/3)$ being present, the dynamics is quite different. Since the initial wealth share of this strategy is small, the dynamics is very similar to the preceding case up to about period 250, see Figure 3 (c). At that time period, the new strategy has gathered about half of the wealth and starts to impact prices. Over the remaining time horizon first the

λ^1 quickly loses wealth to λ^3 and finally the strategy λ^2 is wiped out. The dynamics of prices, Figure 3 (d), differs from the preceding case: the fluctuations die out around time period 400 and prices converge to the values prescribed by strategy λ^3. The investment strategy λ^3 is selected by the market dynamics. Simulations with any number of constant strategies—and the strategy λ^3 being present in the pool—display the same selection outcome in every case tested. This leads to the conjecture of λ^3 being the unique locally evolutionary stable investment strategy in this example.

3.4 The generalized Kelly rule

The task of finding an analogue of the Kelly rule in the case of short-lived assets and incomplete markets is closely related to the search for locally evolutionary stable investment strategies. Of course any such strategy is only a candidate for a *globally* evolutionary stable strategy, but it can be expected that the list of candidates will be short. The above example provides a good motivation and further leads to the conjecture that there is indeed only one such candidate. This claim can be verified as follows.

Suppose there is some strategy λ satisfying (22). Then

$$g_\lambda(\mu) = E \ln \left(\sum_{k=1}^{K} R_k(s) \, \frac{\lambda_k}{\mu_k} \right) \geq E \ln \left(\sum_{k=1}^{K} R_k(s) \, \frac{\mu_k}{\lambda_k} \right)^{-1} > 0$$

by the Jensen inequality because, for every s, $R_k(s)$ is a probability measure on $\{1, ..., K\}$. This finding implies that there is at most one candidate for a locally (and, therefore, globally) evolutionary stable strategy (Hens and Schenk-Hoppé, 2005b, Corollary 1).

Key to the problem of finding the locally evolutionary stable strategy is to achieve an understanding of the properties of the function $\mu \to g_\lambda(\mu)$ with $g_\lambda(\cdot) : \Delta^K \to \mathbb{R}$, defined in (20). The function $g_\lambda(\mu)$ is well-defined for every $\mu \in \Delta^K$ because all components of λ are strictly positive by our assumptions in Section 2.1.

Obviously $g_\lambda(\cdot)$ is a concave function (on $\mathbb{R}^K$) and, by the absence of redundant assets, it is even strictly concave. Therefore, for every given λ there is a unique μ that maximizes $g_\lambda(\mu)$ on the set Δ^K. The quest for a locally evolutionary strategy is the search for a fixed point of this map. The first-order condition for a maximum (which is necessary and sufficient) is given by

$$\sum_{n=1}^{K} E \frac{R_n(s)/\lambda_n}{\sum_{k=1}^{K} R_k(s) \frac{\mu_k}{\lambda_k}} \, \alpha_n = 0$$

for every $\alpha \in \mathbb{R}^K$ with $\sum_{n=1}^{K} \alpha_n = 0$ (because the maximization is over the elements of a simplex). Any fixed point λ^* of the argmax problem therefore solves

$$\sum_{n=1}^{K} E \frac{R_n(s)}{\lambda_n^*} \, \alpha_n = 0$$

because $\sum_{k=1}^{K} R_k(s) = 1$ for each $s = 1, ..., S$. This condition implies

$$\lambda_k^* = ER_k(s) = \sum_{s=1}^{S} \pi(s) R_k(s), \quad k = 1, ..., K \tag{24}$$

The investment strategy (24) is the only candidate for a locally (and, thus, globally) evolutionary stable strategy. $\lambda_k^* > 0$ by the assumptions $\pi(s) > 0$ for $s = 1, ..., S$ and (6). Our analysis shows that λ^* is locally stable against every other constant strategy. With some more work, it can be shown that this results holds true for any stationary strategy. The restriction to strategies that are stationary processes in Δ^K can be justified on grounds of the stationarity of asset payoffs. A more general class would be investment strategies that are adapted, i.e. functions of the (entire) history.

Summarizing the analysis, one can state (Hens and Schenk-Hoppé, 2005b, Theorem 2)

Theorem 3.2. *Suppose the state of the world s_t follows an i.i.d. process. Then the strategy λ^* defined in* (24) *is locally evolutionary stable in every pool of stationary investment strategies.*

If the state of the world follows a Markov process, considerations analogous to the above show that $\lambda_k^*(\hat{s}) = \sum_{s=1}^{S} \pi(s|\hat{s}) R_k(s)$ is the only locally evolutionary stable strategy. In the Markov case, this strategy depends on the current state of nature, and the expectation of the relative payoff $R_k(s)$ is calculated under the transition probabilities, i.e. conditional on the current event. Similar to Theorem 3.2 one needs to impose assumptions on the payoffs and transition probabilities to ensure strict positivity of $\lambda_k^*(\hat{s})$ for all $k = 1, ..., K$.

Interpretation The interpretation of this result is similar to that of the Kelly rule as supplied in Section 1.5. There are some notable exceptions however.

The most striking observation is that the investment strategy λ^* is given by the (conditional) expected value of the relative asset payoffs. This recipe is similar to the Kelly principle of "betting your beliefs" as detailed in Section 1.5. Only the (objective) probabilities and the relative payoffs are needed in the calculation of λ^*. Moreover if the assets are Arrow securities, $R_k(s) \in \{0, 1\}$ and $R_k(s) = 1$ if and only if $k = s$. In this case, $\lambda_{k=s}^* = \pi(s)$ which coincides with the original Kelly rule.

The locally evolutionary stable investment strategy λ^* derived in Theorem 3.2 yields a superior growth rate at its own prices, and it is the only strategy with this property. The result holds in complete as well as in incomplete asset markets, which is remarkable given that a simple analysis using the overtaking criterium does not apply in the latter case. In general however this rule will not maximize the one-period logarithmic growth rate because away from a steady state the composition of the market matters. The wealth distribution and the particular strategies employed by all investors impact the price and thus the log-optimum investment. For Arrow securities λ^* possesses the previously discussed optimality properties. In light of these properties, it is appropriate to call λ^* the (generalized) *Kelly rule* for the short-lived asset market model with incomplete markets.

It might be of interest to inquire whether the Kelly rule λ^* can be linked to utility maximization. Indeed there is a strong connection to logarithmic utility functions in a competitive equilibrium. Suppose prices are given by λ^* and an investor maximizes log utility given these prices (such as in a competitive equilibrium). Then his optimal strategy is λ^*. This is actually part of the reasoning in the proof of Theorem 3.2 which studies a strategies logarithmic growth rate at given prices.

The result can also be interpreted in light of *market selection*. If a λ^* investor is present in the market, this strategy is the only one hat can be selected by the market dynamics. No other investment strategy can gather all the wealth in the market. As explained above, whether selection can occur is related to the performance of a 'mutant strategy' against that of an incumbent: the incumbent's strategy 'sets' prices and the mutant has to play against these prices. This interaction highlights the role of the price mechanism.

3.5 Global dynamics with adaptive strategies

The above result leaves open two questions. First, whether the Kelly rule λ^* is *globally* evolutionarily stable in a pool of stationary strategies as well as for more general payoff matrices. Second, whether this demanding stability property holds true if general, adaptive strategies are permitted. This case is studied in Amir et al. (2005).

In what follows we consider a more general specification of assets where asset k's payoff at time t is given by $A_k(s_t, s_{t-1})$. As above the dependence on s_t is responsible for the randomness of the payoff while the entry s_{t-1}, which is observed at the time of decision making, allows for changes in the payoff structure of the asset. The latter might be caused for instance by the issuer's exposure of the business cycle or through other macroeconomic events. Throughout the remainder of the chapter we will assume that

$$A_k(s, \hat{s}) \geq 0 \quad \text{and} \quad \sum_{k=1}^{K} A_k(s, \hat{s}) > 0 \tag{25}$$

for all k and all $s, \hat{s}$. It is further convenient (and customary) to assume the absence of (conditionally) redundant assets. This condition ensures that different portfolios have different payoff streams and, thus, a unique relationship. For each $\hat{s} = 1, ..., S$, the functions $A_1(\cdot, \hat{s}), ..., A_K(\cdot, \hat{s})$ restricted to the set $\{s \in S : \pi(s \mid \hat{s}) > 0\}$ are linearly independent.

We will impose two additional assumptions. First, the functions

$$R_k^*(\hat{s}) := \sum_{s=1}^{S} \pi(s|\hat{s})\, R_k(s, \hat{s}) = E[R_k(s_{t+1}, s_t) \mid s_t = \hat{s}] \tag{26}$$

$k = 1, 2, ..., K$, take on strictly positive values for each $\hat{s} = 1, ..., S$. (26) is the conditional expectation of the relative payoff of every asset k given $s_t = \hat{s}$. Second, condition (8) is tightened. The coordinates $\lambda_{k,t}(s^t)$ of every investment strategy are bounded away from zero by a non-random constant $\gamma > 0$, i.e. $\inf_{i,k,t,s^t} \lambda^i_{k,t}(s^t) \geq \gamma > 0$. The constant γ might depend on the strategy λ, but not on k, t and s^t.

As for payoff functions $A(s)$, the condition on the absence of redundant assets for $A(s,\hat{s})$ implies the same property for the relative payoff functions $R(s,\hat{s})$. Therefore the assumption (A.2) in Amir et al. (2005) is satisfied.

The Kelly rule is defined as a function of the conditional expectation of the relative payoffs (26):

$$\lambda^*_{k,t}(s_t) = R^*_k(s_t) \tag{27}$$

The dynamics of relative wealth of investment strategies is given by

$$r^i_{t+1} = \sum_{k=1}^{K} R_k(s_{t+1}, s_t) \frac{\lambda^i_{k,t}(s^t) r^i_t}{\langle \lambda_{k,t}(s^t), r_t \rangle} \tag{28}$$

where, cf. (17),

$$R_k(s,\hat{s}) = \frac{A_k(s,\hat{s})}{\sum_{n=1}^{K} A_n(s,\hat{s})}$$

The availability of general adaptive strategies enable investors to buy the market portfolio which, in the model (28), entails a payoff equal to the invested wealth. An investment strategy always buying the market portfolio will therefore possess constant wealth share (equal to its initial fortune).

Suppose investment strategies $\lambda^i_t(s^t)$, $i = 2, ..., I$, are given. We can then define an adapted investment strategy for investor 1 by

$$\lambda^1_{k,t} = \frac{1}{1 - r^1_t} \sum_{j=2}^{I} \lambda^j_{k,t}\, r^j_t \tag{29}$$

This strategy's portfolio shares are equal to the price of each asset because $\lambda^1_{k,t} = \sum_{j=1}^{I} \lambda^j_{k,t}\, r^j_t = q_{k,t}$. According to (28), the wealth dynamics of the investment strategy $\lambda^1_{k,t}$ is

$$r^1_{t+1} = \sum_{k=1}^{K} R_k(s_{t+1}, s_t) \frac{\lambda^1_{k,t} r^1_t}{\langle \lambda_{k,t}, r_t \rangle} = \sum_{k=1}^{K} R_k(s_{t+1}, s_t)\, r^1_t = r^1_t \tag{30}$$

Investor 1's wealth share remains constant over time (regardless of the states of the world revealed). The portfolio positions can be equated as

$$\theta^1_{k,t} = \frac{\lambda^1_{k,t} r^1_t}{\sum_{j=1}^{I} \lambda^j_{k,t}\, r^j_t} = r^1_t \tag{31}$$

which means the portfolio θ^1_t is proportional to the market portfolio (or, equivalently, the total supply) which is given by $(1, ..., 1)$.

These considerations highlight the importance of the market portfolio in this modeling framework. It also follows that market selection can only occur if none of the investment strategies (asymptotically) coincides with the market portfolio. The market portfolio in this model provides a protection against extinction.

A 'virtual' investment strategy $\zeta_t = (\zeta_{1,t}, ..., \zeta_{K,t})$ which would lead to the market portfolio can be defined through (29):

$$\zeta_{k,t} = \frac{1}{1 - r_t^1} \sum_{j=2}^{I} \lambda_{k,t}^j \, r_t^j \tag{32}$$

The surprising result (Amir et al., 2005, Theorem 1) is that the Kelly strategy is selected by the market dynamics if it stays asymptotically distinct from the market portfolio. In other words, if investor 1 uses the Kelly rule, while all the others use strategies distinct from the Kelly rule and the Kelly rule does not converge to the market portfolio, investor 1 is almost surely the *single survivor* in the market selection process.

Theorem 3.3. *Let investor 1 use the Kelly strategy $\lambda^1 = \lambda^*$ defined by* (27)*. Suppose with probability 1, one has*

$$\liminf_{t \to \infty} |\lambda^*(s_t) - \zeta_t| > 0 \tag{33}$$

Then Kelly investor 1 is a single survivor. Moreover,

$$\liminf_{t \to \infty} \frac{1}{t} \ln \frac{r_t^1}{1 - r_t^1} > 0 \tag{34}$$

almost surely.

The symbol $|\cdot|$ denotes the sum of the absolute values of the coordinates of a finite-dimensional vector.

The convergence property of the Kelly investor's wealth share means that it tends to one at an exponential rate while the wealth share of the other investors vanishes at the same rate. The strategy λ^* dominates the other investors exponentially.

In the case of constant strategies, payoffs $R(s_{t+1})$ and i.i.d. states of the world, Theorem 3.3 makes the identical statement as the result (Evstigneev et al., 2002, Theorem 3.1) which says that if investor 1 uses the Kelly rule $\lambda^1 = \lambda^* = \sum_{s=1}^{S} \pi(s) R_k(s)$, while all the other investors $j \geq 2$ use constant strategies $\lambda^J \neq \lambda^*$. Then investor 1 is the single survivor. The proof relies on the (non-trivial) observation that

$$E \ln \sum_{k=1}^{K} R_k(s) \frac{\lambda_k^*}{\lambda_k^* \, r + \mu_k \, (1 - r)} > E \ln \sum_{k=1}^{K} R_k(s) \frac{\mu_k}{\lambda_k^* \, r + \mu_k \, (1 - r)}$$

for any $\mu \in \Delta^K$ with $\mu > 0$ and $\lambda^* \neq \mu$, and any $r \in [0, 1]$, cf. (Evstigneev et al., 2002, Lemma 3.1). This result asserts the superiority of the Kelly investor's growth rate per unit invested.

The general case covered in Theorem 3.3 rests on a similar property for the conditional expected value.

4 An evolutionary stock market model

This section introduces an evolutionary finance model of a stock market. This framework overcomes the main shortcoming of the model with short-lived assets which is

discussed in the preceding sections. Whereas short-lived assets pay off and disappear and new assets have to be issued in each period, a stock in a company entitles its holder to a (risky) payoff stream. Stocks can experience capital gains and/or losses. The availability of such a model is of particular importance for applications to real markets. The following is based on Evstigneev et al. (2006, 2008). This model has been used to study Tobin's liquidity preference argument from an evolutionary perspective in Hens and Schenk-Hoppé (2006). An application to insurance markets (in which liquidity shocks are present) is discussed in De Giorgi (2008).

The main difference to the previous model is that assets are issued at time zero and 'live' forever. In each period in time, these long-lived assets have a market price and can be traded among all investment strategies. While short-lived assets paid in terms of wealth which was spend on new assets, long-lived assets require a different approach. The idea of Lucas (1978) is applied: the asset pay off in units of a perishable consumption good (whose price is also taken as the numeraire). Consumption will be modeled through a common consumption rate to provide a level playing field for the investment strategies as it separates the investment and consumption decisions. In the context of stocks these payoffs can be interpreted as dividends payments. This specification of asset payoffs implies that investment strategies with above average dividend income (relative to their wealth) will sell dividends in exchange for assets to the under performing investment strategies. It turns out that the model with short-lived assets can be accommodated by a particular choice of the consumption rate.

The model is derived step-by-step. We start with a simple accounting identity linking two successive periods in time. An investment strategy's wealth in period $t+1$ is derived from this strategy's portfolio holdings $\theta^i_{k,t}$, the realized asset payoffs $A_k(s_{t+1})$ and the resale prices of assets $pk, t+1$. One has

$$w^i_{t+1} = \sum_{k=1}^{K} \big(A_k(s_{t+1}) + p_{k,t+1}\big)\, \theta^i_{k,t} \tag{35}$$

For shortness the notation $\lambda_t = \lambda_t(s^t)$ is used in the following. Inserting (9) and (10) in (35), one obtains the dynamics

$$w^i_{t+1} = \sum_{k=1}^{K} \big(A_k(s_{t+1}) + \langle \lambda_{k,t+1}, b_{t+1}\rangle\big) \frac{\lambda^i_{k,t}\, b^i_t}{\langle \lambda_{k,t}, b_t\rangle} \tag{36}$$

where the budget b^i_t is defined by a strategy's saving rate and current wealth.

If all the investment strategies have a common saving rate ρ, the budgets are given by $b^i_t = \rho w^i_t$. Then (36) takes the form

$$w^i_{t+1} = \sum_{k=1}^{K} \big(A_k(s_{t+1}) + \rho\langle \lambda_{k,t+1}, w_{t+1}\rangle\big) \frac{\lambda^i_{k,t}\, w^i_t}{\langle \lambda_{k,t}, w_t\rangle} \tag{37}$$

The aggregate wealth $W_{t+1} = \sum_i w^i_{t+1}$ can be equated as (summation of (36) over i)

$$\begin{aligned} W_{t+1} &= \sum_{k=1}^{K} \left(A_k(s_{t+1}) + \langle \lambda_{k,t+1}, b_{t+1}\rangle\right) \left[\sum_{i=1}^{I} \frac{\lambda^i_{k,t}\, b^i_t}{\langle \lambda_{k,t}, b_t\rangle}\right] \\ &= A_{t+1} + \rho \sum_{j=1}^{I} \left[\sum_{k=1}^{K} \lambda^j_{k,t+1}\right] w^j_{t+1} = A_{t+1} + \rho W_{t+1} \end{aligned} \tag{38}$$

with $A_{t+1} = \sum_{k=1}^{K} A_k(s_{t+1})$. One finds

$$W_{t+1} = \frac{A_{t+1}}{1-\rho} \tag{39}$$

This finding in particular implies that the aggregate expenditure (demand) for the consumption good is equal to the value of the aggregate supply, $(1-\rho)W_{t+1} = A_{t+1}$. This is Walras' law: the market for each asset clears, investors exhaust their budgets and, thus, the market for the consumption good clears as well. This consideration shows that the price of consumption good is set to one; no price variable is placed in front of the payoffs A_k in (36).

Employing (39), a relation for the investors' wealth shares $r^i_t = w^i_t/W_t$ can be obtained

$$r^i_{t+1} = \sum_{k=1}^{K} \left((1-\rho)R_k(s_{t+1}) + \rho\langle \lambda_{k,t+1}, r_{t+1}\rangle\right) \frac{\lambda^i_{k,t}\, r^i_t}{\langle \lambda_{k,t}, r_t\rangle} \tag{40}$$

$i = 1, ..., I$. Recall that

$$R_k(s) = \frac{A_k(s)}{\sum_{n=1}^{K} A_n(s)}$$

The system (41) is linear in the vector r_t and can be written in matrix notation. Let $\lambda_{k,t} = (\lambda^1_{k,t}, ..., \lambda^I_{k,t})$ and denote by $\Lambda^T_t = (\lambda^T_{1,t}, ..., \lambda^T_{K,t}) \in \mathbb{R}^{I\times K}$ the matrix of investment strategies. $\Theta_t \in \mathbb{R}^{I\times K}$ is the matrix of portfolios and $R(s_t)^T = (R_1(s_t), ..., R_I(s_t)) \in \mathbb{R}^I$ is the vector of dividend payments in period t. Then (41) can be written as

$$r_{t+1} = (1-\rho)\Theta_t R(s_{t+1}) + \rho\Theta_t\Lambda_{t+1} r_{t+1} \tag{41}$$

this equation is equivalent to

$$r_{t+1} = (1-\rho)\left[\mathrm{Id} - \rho\Theta_t\Lambda_{t+1}\right]^{-1}\Theta_t R(s_{t+1}) \tag{42}$$

The last step requires the existence of the inverse of the matrix $\mathrm{Id} - \rho\Theta_t\Lambda_{t+1}$. This is ensured by the fact that the matrix is a contraction for every $0 \le \rho < 1$, cf. (Evstigneev et al., 2008, Proposition 1).

Remark. Setting the saving rate $\rho = 0$ one obtains the evolutionary finance model with short-lived assets, cf. (28). This observation leads to a comprehensive interpretation of the components of (42). $\Theta_t R(s_{t+1})$ gives the investment strategies' dividend gains, i.e. income from asset payoffs, while $\left[\mathrm{Id} - \rho\Theta_t\Lambda_{t+1}\right]^{-1}$ are the capital gains, i.e. changes in the book value of asset holdings due to changes in asset prices. The factor $(1-\rho)$ stems from the normalization to express wealth in terms of investors' shares of the total wealth.

An alternative and computationally efficient method to solve (40) is to determine the prices $q_{t+1,k} = \langle \lambda_{t+1,k}, r_{t+1} \rangle$ first. Then these prices are inserted on the right-hand side of (40) and the vector of wealth shares r_{t+1} can easily be calculated. Rather than deriving the inverse of a matrix with dimension $I \times I$, one only needs to invert a (typically much smaller) $K \times K$ matrix. (40) gives

$$q_{t+1} = (1-\rho)\left[\mathrm{Id} - \rho\Lambda_{t+1}\Theta_t\right]^{-1}\Lambda_{t+1}\Theta_t R(s_{t+1}) \tag{43}$$

where Id is the $K \times K$-dimensional identity matrix.

The dynamics of the investment strategies' wealth shares has several features in common with those of the evolutionary finance model with short-lived assets. Every vertex of the simplex Δ^K is fixed point, the faces are invariant, the interior of the simplex (and each face) is invariant nd there are no deterministic fixed points in the interior of Δ^K.

4.1 Local dynamics

The analysis of the long-term dynamics of the evolutionary finance model with long-lived assets is similar to the case of short-lived assets considered in Section 3.2. There are, however, several interesting features that are unique to this model. These properties will only surface for strategies that are stationary and time-variant rather than constant. In this section it is assumed that strategies can depend on the past, i.e. for each i, $\lambda_t^i = \lambda^i(s^t)$. Again we provide a heuristic analysis of an investment strategy's growth rate close to a fixed point, see Evstigneev et al. (2006) for a mathematically precise derivation.

Suppose $r_t \approx e^i$ for all t. Then the price $q_{k,t} \approx \lambda_k^i$ and the dynamics of strategy j's wealth share can be approximated by (see (40))

$$r_{t+1}^j \approx \left[\sum_{k=1}^K \frac{(1-\rho)R_k(s_{t+1}) + \rho\lambda_{k,t+1}^i}{\lambda_{k,t}^i}\,\lambda_{k,t}^j\right] r_t^j \tag{44}$$

which implies an approximate logarithmic growth rate

$$\frac{1}{t}\ln\left(r_t^j/r_0^j\right) \underset{t\to\infty}{\longrightarrow} E\ln\left(\sum_{k=1}^K \frac{(1-\rho)R_k(s_1) + \rho\lambda_k^i(s^1)}{\lambda_k^i(s^0)}\,\lambda_k^j(s^0)\right) =: g_{\lambda^i}(\lambda^j) \tag{45}$$

Suppose the state of the world follows an i.i.d. process. Then a constant strategy of incumbent i will induce an i.i.d. returns process and additionally there are no capital gains. This case can be studied completely analogous to Section 3.2. Define $\lambda_k^* = \sum_{s=1}^S \pi(s)R_k(s)$ as in (24). Then (45) gives

$$\begin{aligned} g_{\lambda^*}(\lambda^j) &= E\ln\left(\rho + (1-\rho)\sum_{k=1}^K \frac{R_k(s_1)}{\lambda_k^*}\lambda_k^j(s^0)\right) \\ &= \int_{S^{\mathbb{N}}} \sum_{s=1}^S \pi(s)\ln\left(\rho + (1-\rho)\sum_{k=1}^K \frac{R_k(s)}{\lambda_k^*}\lambda_k^j(s^0)\right) dP^0(s^0) \end{aligned}$$

where P^0 denotes the probability distribution for the histories s^0. For each fixed history s^0, the inner term is strictly negative if $\lambda^j(s^0) \neq \lambda^*$ and zero if both coincide, cf. Section 3.2. Therefore λ^* is locally evolutionary stable against all stationary investment strategies.

If the state of the world follows a Markov process with transition probability $\pi(\cdot|\cdot)$, then (45) can be written as

$$g_{\lambda^i}(\lambda^j) = \int_{S^{\mathbb{N}}} \tilde{g}_{\lambda^i}(\lambda^j, s^0) dP^0(s^0) \tag{46}$$

with

$$\tilde{g}_{\lambda^i}(\lambda^j, s^0) = \sum_{s=1}^{S} \pi(s_0|s_1) \ln \left(\sum_{k=1}^{K} \frac{(1-\rho)R_k(s_1) + \rho\lambda_k^i(s^1)}{\lambda_k^i(s^0)} \lambda_k^j(s^0) \right)$$

being the expected logarithmic growth rate of strategy λ^j at λ^i prices for a given history s^0.

Even if the incumbent's strategy is constant, returns will follow a Markov process and evolutionary stability will fail. On the other hand, if the incumbent has a Markov strategy, then the returns are Markov as well. Indeed it turns out locally evolutionary stable strategies are Markov.

For the analysis of local stability of a stationary strategy μ in a market with λ-price system, it suffices to study the integrant $\tilde{g}_\lambda(\mu, s^0)$ in (46). If this term is non-negative and strictly negative on a set of histories of positive measure, $g_\lambda(\mu) < 0$. A maximum is obtained at $\mu = \lambda$ if the first-order condition holds

$$\sum_{n=1}^{K} \left(\frac{\partial \tilde{g}_\lambda(\mu, s^0)}{\partial \mu_n} \bigg|_{\mu=\lambda} \right) \alpha_n = \sum_{n=1}^{K} \sum_{s=1}^{S} \pi(s_0|s_1) \frac{(1-\rho)\, R_n(s_1) + \rho\lambda_n(s^1)}{\lambda_n(s^0)} \alpha_n = 0$$

for every $\alpha \in \mathbb{R}^K$ with $\sum_{n=1}^{K} \alpha_n = 0$. This implies that the conditional expected return of each asset must be constant, i.e.

$$\sum_{s=1}^{S} \pi(s_0|s_1) \frac{(1-\rho)\, R_n(s_1) + \rho\lambda_n(s^1)}{\lambda_n(s^0)} = const.$$

It is not too hard to see that the only investment strategy with this property is given by the function $\lambda^* : S \times K \to [0, 1]$ defined as

$$\lambda^* = \frac{1-\rho}{\rho} \sum_{t=1}^{\infty} \rho^t \pi^t R, \tag{47}$$

where $\pi^t = \pi \dots \pi$ denotes the t-period transition probability with $\pi_{s\tilde{s}} = \pi(s|\tilde{s})$. The investment strategy (47) will be referred to as the *Kelly rule* for reasons explained in detail below.

The local stability resp. instability of an investment strategy might not be determined by the first-order condition if this strategy is stationary rather than just constant. The condition is only sufficient is the rank of the $K \times S$-dimensional matrix of returns (with elements $(1-\rho)R_k(s) + \rho\lambda_k^*(s)$) is equal to K.

That the strategy λ^* is locally stable against all stationary investment strategies, i.e. $g_{\lambda^*}(\mu) < 0$ for all μ such that $\mu(s^0) \neq \lambda^*(s^0)$ on a set of positive measure, can be seen as follows. At the prices λ^* given by (47), the return matrix has full rank. One has

$$(1-\rho)R + \rho\lambda^* = (1-\rho)R + (1-\rho)\sum_{t=1}^{\infty}\rho^t\pi^t R = (1-\rho)\sum_{t=0}^{\infty}\rho^t\pi^t R = (1-\rho)[\mathrm{Id}-\rho\pi]^{-1}R$$

The inverse of $\mathrm{Id} - \rho\pi$ is well-defined because $[\mathrm{Id} - \rho\pi]x = 0 \iff x = \rho\pi x$ and $\rho\pi$ is a contraction. Since R has full rank by assumption, the above relation implies that the matrix of returns has full rank.

If the incumbent pursues a strategy different from the Kelly rule (47) one can construct strategies that have a strictly positive growth rate.

Summarizing, one has (Evstigneev et al., 2006, Theorem 1)

Theorem 4.1. *The investment strategy λ^* defined in* (47) *is the only locally stable investment strategy. That is for each stationary strategy $\mu \neq \lambda^*$ one has (a) $g_{\lambda^*}(\mu) < 0$ and (b) there exists a stationary investment strategy λ such that $g_\lambda(\mu) > 0$.*

Interpretation The investment strategy defined in (47) derives its portfolio shares from the fundamental value assets. For a given state of the world, the term on the right-hand side of (47) is the discounted expected relative payoff of each of the assets. The discount factor is given by the saving rate and the expected value is calculated with respect to the conditional expectation. As in the case of short-lived assets, the relative payoff of an asset is important, not the absolute payoff.

The investment strategy (47) merits the term Kelly rule because it is a natural extension of 'betting your beliefs' to the framework of long-lived assets with Markov state of the world. All that is needed in the calculation of (47) are the transition probabilities and the asset payoffs. If the state of the world is an i.i.d. process, then $\pi^t = \pi$ and, therefore, (47) collapses to $\sum_s \pi(s)R_k(s)$. For Arrow securities this investment strategy coincides with the Kelly rule in betting markets: $\lambda^*_k = \pi(k)$, see Section 1.5.

The result shows that the only locally evolutionary stable investment strategy is the Kelly rule (47). A market in which a Kelly investor is the incumbent, relative asset prices are given by their fundamental value in terms of their relative payoffs. The robustness of this market against any stationary mutant strategy implies that deviations from the fundamental relative valuation are corrected over time. This finding provides a novel asset pricing hypothesis for dividend-bearing assets such as stocks traded on security exchanges, see Section 5.3.

4.2 Global dynamics with constant strategies

The global dynamics of the evolutionary finance model with long-lived asset is considerably more demanding to analyze than the short-lived asset case. At present, the wealth dynamics of a market in which a Kelly investor is present is only fully understood when all investment strategies are constant and the state of the world is governed

by an i.i.d. process. The following briefly summarizes the main findings obtained in Evstigneev et al. (2008).

Define the constant investment strategy $\lambda^* = (\lambda_1^*, ..., \lambda_K^*)$ by

$$\lambda_k^* = ER_k(s) = \sum_{s=1}^{S} \pi(s) R_k(s) \tag{48}$$

for $k = 1, ..., K$. Each budget share λ_k^* is the *expected relative dividend* of the respective asset.

To formulate the main result, a couple of definitions are required. An investment strategy $\lambda^i = (\lambda_1^i, ..., \lambda_K^i)$ *survives* with probability one if $\lim_{t\to\infty} r_t^i > 0$ almost surely. It *becomes extinct* with probability one if $\lim_{t\to\infty} r_t^i = 0$ almost surely. The investment strategy $\lambda = (\lambda_1, ..., \lambda_K)$ is called *globally evolutionary stable* if the following condition holds. Suppose investor 1 uses the strategy λ, while all the others investors $j = 2, ..., I$ use portfolio rules $\hat{\lambda}^j$ distinct from λ, then investor 1 survives with probability one, whereas all the other investors become extinct with probability one.

One has (Evstigneev et al., 2008, Theorem 1)

Theorem 4.2. *The Kelly investment strategy λ^* defined in* (48) *is globally evolutionary stable in the pool of constant strategies.*

The strategy λ^* can be interpreted as a generalization of the Kelly rule because, in the case of Arrow securities, the portfolio shares λ_k^* are equal to the probability of the corresponding state of the world. The presence of a price dynamics (which implies the potential for capital gains and losses) however highlights the quite remarkable nature of the result in Theorem 4.2. Details are given after a brief discussion of the proof.

The proof of this result relies on the observation that the Kelly investor's wealth share has a positive expected logarithmic return. This growth rate is strictly positive if and only if the current prices do not coincide with the Kelly rule. In formal notation this statement can be expressed as follows. Let r be the distribution of wealth shares across investment strategies at some period in time t. Then the asset prices in period t are given by $p_k = \langle \lambda_k, r \rangle$. The solution to

$$F^i(s, r) = \sum_{k=1}^{K} \big(\rho \langle \lambda_k, F(s, r)\rangle + (1-\rho) R_k(s)\big) \frac{\lambda_k^i r^i}{p_k},\ i = 1, ..., I, \tag{49}$$

which corresponds to (40), defines the asset prices in the subsequent period in time

$$q_k(s) = \langle \lambda_k, F(s, r)\rangle$$

Theorem 3 in Evstigneev et al. (2008) asserts that for each $r \in \Delta^I$ one has

$$E \ln \left(\sum_{k=1}^{K} \frac{\rho q_k(s) + (1-\rho) R_k(s)}{p_k} \lambda_k^* \right) \geq 0, \tag{50}$$

with strict inequality if and only if $p_k \neq \lambda_k^*$ for at least one $k = 1, ..., K$.

A related result is employed in the analysis of the global dynamics of constant investment strategies for short-lived assets, see the discussion towards the end of Section 3.5. The impact of the price dynamics, which stems from the appearance of the prices $q_k(s)$ in (50), considerably raises the level of difficulty in studying the growth rates of investment strategies.

The case of adapted investment strategies and a Markovian dividend process is still open.

Interpretation The above result has the following interpretation and several interesting implications.

Theorem 4.2 states that if all investors are constrained by being required to choose constant investment strategies, there is exactly one strategy that will do best in the long term. It is the rule that divides an investor's wealth in proportions given by the expected relative dividends. The same investment strategy was discovered in the case of short-lived assets, Section 3.5. In the present case however assets are long-lived and there is a price dynamics. The variations in the asset prices entails capital gains (and losses) in the investors' portfolio holdings, which is absent for short-lived assets. It is therefore not obvious whether a constant investment strategy can be globally evolutionary stable. Convergence of the wealth dynamics moreover implies the non-stationary of prices. These observations show the depths of the finding in Theorem 4.2.

Referring to λ^* as a generalization of the Kelly rule has some justification (as explained above). Most of the features the Kelly rule possesses in betting markets though do not carry over to the stock market model. Indeed only two features are preserved: the form (dividend payoffs and expected value) and the property of gathering all the wealth in the long run.

This investment strategy λ^* does not match the growth optimal portfolio in general. The former is constant while the latter would depend on the price process and, thus, vary over time. The important exception is the case in which asset prices are constant and equal λ^*_k. Then the investment strategy λ^* maximizes the expected logarithmic growth rate (see (50)). This implies that all strategies different to λ^*_k will have negative growth rates.

The growth optimality of λ^* at its 'own' prices has been observed as well in the local analysis, Section 4.1. Indeed this observation confirms that the linearized and the actual dynamics have the same qualitative properties close to the steady state in which the λ^* investor owns all wealth.

The long-term success of the strategy λ^* is rooted in another property as well. If prices are not equal to the vector λ^*, the prices dynamics is not trivial as it is driven by the wealth dynamics in the pool of strategies present in the market. In these circumstances it is the (expected logarithmic) growth rate of a λ^* investor's wealth share that matters for the long-term dynamics. This property is at the heart of the proof of Theorem 4.2. Eq. (50)) ensures that the λ^* investor's relative wealth will, on average, grow: the investor's logarithmic growth rate is strictly positive if the current asset prices do not match λ^*. A positive growth rate can be interpreted as experiencing faster growth than the 'average investor.' It is straightforward from (50) that an investment strategy that is equal to the current prices has a growth rate

equal to zero because $\sum_{k=1}^{K} \rho q_k(s) + (1-\rho)R_k(s) = 1$ for every state of nature and every price vector $q(s)$. The positive growth rate of a λ^* investor's wealth is surprising because prices do vary over time.

Theorem 4.2 is a deep result in that it shows that the price dynamics induced in a pool of constant investment strategies (and i.i.d. dividend payoffs) favors a λ^* investor for *every* distribution of wealth shares. The above-average expected growth of the λ^* investor's wealth holds in every period in time and for every current price system. The asset prices in the subsequent period in time however are tied down by the wealth dynamics and, due to the investment strategies being constant, the possible outcomes of the price vector are linked to the random payoffs of the assets. It is important to emphasize that these price dynamics is non-stationary since prices converge.

The mechanism behind this growth stems from the fact that a λ^* investor holds more of those assets with a price lower than their expected relative dividend and fewer of those with prices exceeding λ_k^*. Viewing λ_k^* as benchmark, these positions can be characterized as being long resp. short in relative terms. The potential capital gains/losses caused by the other investors' strategies and the wealth dynamics do not have a systematic negative effect. The positivity of the λ^* investor's growth rate means he has excess returns, i.e. his logarithmic return is higher than the market average (which is given by the prices). The asset prices eventually converge to the λ^* benchmark because the λ^* investor will gradually increase his share of the total wealth because his expected logarithmic growth rate of relative wealth is positive.

It is obvious how to leverage this result. Identifying assets that are underpriced resp. overpriced relative to the λ^* benchmark, one could construct a self-financing portfolio by going long resp. short in these assets. This should potentially boost the growth rate, but, on the other hand, increases the risk. Bankruptcy, which is absent in our framework because the λ^* investor only has long positions, becomes a real risk.

Surprisingly, perhaps, this investment advice is not new. It can be traced back at least to Graham and Dodd (1934) who claimed that excess returns can be reaped from the tendency of markets to converge towards fundamental values. Our approach provides a formal model to support this claim which is derived from empirical observations.

4.3 Kelly rule in general equilibrium

This section describes the Kelly rule as an outcome of optimal investment and consumption behavior within a dynamic general equilibrium model in which agents have perfect foresight. This result is of interest because this framework is standard in the asset pricing literature (as well as being the foundation for most of dynamic macroeconomics). The equilibrium concept goes back to Radner (1972) who called it an equilibrium in plans, prices and price expectations. In such an equilibrium every current decision requires the knowledge of the result of all decisions in the future. This is the exact opposite of the approach followed in our evolutionary models. In evolutionary finance only historical observations influence current behavior; no agreement about the future events is required. Time moves forward—in the sense of dynamical systems—in contrast to the simultaneity of past, presence and future in general equilibrium.

Assume the state of the world follows an i.i.d. process and asset payoffs at time t are given by $A_k(s_t)$, $k = 1, ..., K$. The stochastic structure modeling uncertainty about future states is identical to that in the evolutionary models.

The plan of agent i is given by a consumption-investment process (ρ^i, λ^i) with a saving rate process $\rho^i = (\rho^i_t)$ and an investment strategy $\lambda^i = (\lambda^i_t)$, $t \geq 0$. A price system is a process $p = (p_t)$, $t \geq 0$ with $p_t(s^t) \in \mathbb{R}^K_{++}$. All processes have to be adapted to the filtration generated by the i.i.d. state of the world. Given a price system p and a plan (ρ^i, λ^i), the wealth w^i_t of agent i evolves as (cf. (12))

$$w^i_{t+1} = \left(\sum_{k=1}^{K} \frac{A_k(s_{t+1}) + p_{k,t+1}}{p_{k,t}} \lambda^i_{k,t} \right) \rho^i_t w^i_t \tag{51}$$

Each agent maximizes, for given price process p, the expected discounted logarithmic utility from consumption

$$U^i = \mathbb{E} \sum_{t=0}^{\infty} (\beta^i)^t \ln(c^i_t) \tag{52}$$

with consumption given by $c^i_t = (1 - \rho^i_t) w^i_t$. The discount factor is $0 < \beta^i < 1$.

An equilibrium is given by a price process p and plans (ρ^i, λ^i), $i = 1, ..., I$ such that (a) the plans are optimal for the price process p, i.e. maximize (52); and (b) markets clear, i.e.

$$p_{k,t} = \sum_i \lambda^i_{k,t} (1 - \rho^i_t) w^i_t$$

for the plans (ρ^i, λ^i), $i = 1, ..., I$.

One has the following result:

Theorem 4.3. *The above dynamic general equilibrium model has a competitive equilibrium in which each agent's optimal investment strategy is given by λ^*.*

Let us given some intuition for the proof of this result which is related to Gerber et al. (2007). They consider a version with more general utility functions but in a model with finite-time horizon. Rewrite (51) as

$$w^i_{t+1} = w^i_0 \prod_{u=0}^{t} \rho^i_u \sum_{k=1}^{K} \frac{A_k(s_{u+1}) + p_{k,u+1}}{p_{k,u}} \lambda^i_{k,u} \tag{53}$$

for $t \geq 1$.

The first-order conditions for the saving rate process show that $\rho^i_t \equiv \beta^i$. Optimality of the investment strategy is derived from the first-order condition for $\lambda^i_{n,t}$:

$$E_t \sum_{u=t+1}^{\infty} (\beta^i)^u \frac{[A_n(s_{u+1}) + p_{n,u+1}]/p_{n,u}}{\sum_{k=1}^{K} [A_k(s_{u+1}) + p_{k,u+1}] \lambda^i_{k,u} / p_{k,u}} = \xi^i_t \tag{54}$$

where ξ^i_t is the Lagrange multiplier corresponding to the constraint $\sum_k \lambda^i_{k,t} = 1$.

The equilibrium specification in Theorem 4.3 gives $p_{k,u} = \lambda^*_k \sum_i (1 - \beta^i) w^i_u$ and $\lambda^i_{k,u} \equiv \lambda^*_k = E[A_k(s) / \sum_n A_n(s)]$. Since all agents follow the same strategy, one

further has $w_u^i = (\beta^i/\beta^1)^u (w_0^i/w_0^1) w_u^1$. With these observations one can prove that the left-hand side of (54) is independent of n, which shows optimality of the Kelly rule in this equilibrium.

Finally the transversality condition needs to be verified. Logarithmic utility makes this a straightforward task:

$$\lim_{t\to\infty} (\beta^i)^t \frac{\partial \ln(c_t^i)/\partial c_t^i}{\partial \ln(c_0^i)/\partial c_0^i} \, c_t^i = \lim_{t\to\infty} (\beta^i)^t c_0^i = 0$$

The proof of Theorem 4.3 implies a certain uniqueness property of the equilibrium: If all investors pursue the same strategy λ and if this strategy is constant, then $\lambda = \lambda^*$.

5 Applications

This section discusses a range of applications of evolutionary finance theory. Numerical simulations of the evolutionary finance models introduced in Sections 3 and 4 allow the study of a variety of applied issues: dynamics of asset prices, long-term asset pricing benchmarks, performance of agent-based portfolio choice, co-existence of investment strategies.

All of the following studies use the same set of dividend data. The asset payoffs are modeled by the (annualized) dividends paid by firms in the Dow Jones Industrial Average index (DJIA) during the 26 year period 1981-2006. The data are obtained from the CRSP database. Each year is associated with a particular state of the world which is drawn according to an i.i.d. process distributed uniformly across the 26 potential outcomes. Related studies are Hens and Schenk-Hoppé (2004) and Hens et al. (2002).

The following subsections present three topics. A simulation analysis of the wealth dynamics of a large set of common investment strategies (and the Kelly rule), Section 5.1. The possibility of the evolution of the Kelly rule (rather than being present in a market from the beginning) in a framework in which the set of strategies is not a priori fixed but their evolution is modeled by genetic programming with tournament selection, Section 5.2. An empirical test of the predictions of evolutionary finance on asset pricing and the convergence of prices, which is closely related to the value premium puzzle, Section 5.3.

5.1 Simulation studies

The numerical study is based on the dividend data of all firms that have been listed without interruption in the DJIA during the period 1981-2006. There are $K = 16$ firms with this property. Denote by $D(s) \in R_+^K$ the vector of firms' total dividend payment in year $s + 1980$, $s = 1, ..., 26$ is the state of the world. Define the relative dividend of firm k paid in state s by

$$d_k(s) = \frac{D_k(s)}{\sum_{n=1}^K D_n(s)}$$

Sample paths of firms' relative dividend payment are obtained by random draws from the set $\{1, ..., 26\}$ using a uniformly distributed i.i.d. process. This generates samples of infinite length by "randomizing the years."

Myopic mean-variance optimization Agent-based models often assume investors who plan just one period ahead and maximize a CARA utility function. This specification is prominent in the noise trader literature (De Long et al. (1990)), see also Hommes (2001); Hommes and Wagener (2009). The evolutionary finance framework enables an assessment of the robustness of markets (with the above real-world background) in which myopic mean-variance traders are present. The dynamics is described by the evolutionary finance model with long-lived assets, Section 4. There are only two investment strategies present in the market. A mean-variance optimizer who takes into account the statistics of the dividend process as well as the prices that will prevail in the long-term. The other investment strategy corresponds to an investor who is a victim of illusionary diversification and distributes his wealth equally across assets. In both cases all investors have constant strategies. Two cases of mean-variance maximizers are considered: The global minimum-variance portfolio, Figure 4, and the tangency portfolio with net interest rate set to zero, Figure 5.

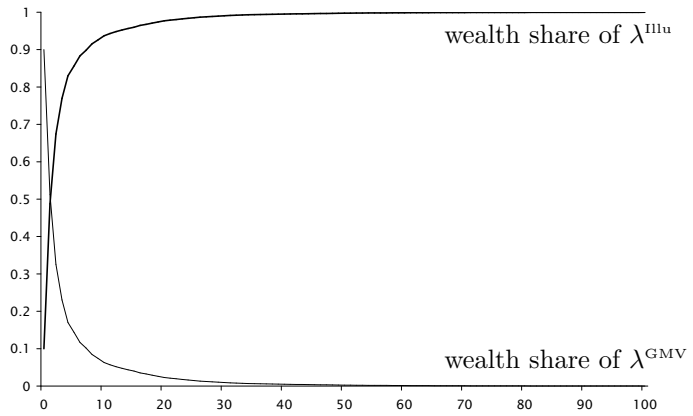


Figure 4: Dynamics of wealth shares in a market with an illusionary diversification strategy λ^{Illu} and the globally minimum mean-variance rule λ^{GMV}.

The wealth dynamics depicted in Figures 4–5 illustrates the surprisingly poor performance of mean-variance optimization in competition with a rather unsophisticated investment strategy. The simple-minded investor following the illusionary diversification rule drives out a globally minimum mean-variance investor as well as the holder of the tangent portfolio. The figures show typical runs of the disadvantaged λ^{Illu}-investor who is only endowed with 10% of the wealth. These findings highlight the importance of studying financial market dynamics outside a mean-variance framework. We consider the simulation results as a major challenge to the literature on agent-based

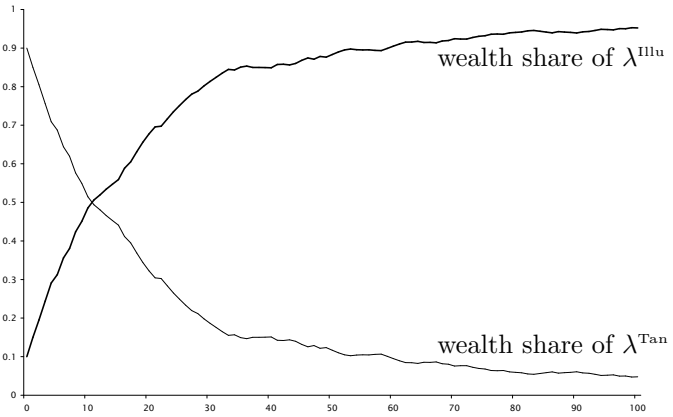


Figure 5: Dynamics of wealth shares in a market with an illusionary diversification strategy λ^{Illu} and the tangency mean-variance rule λ^{Tan}.

modeling with mean-variance investors.

The performance of adaptive strategies The above study can be placed in a much broader context by increasing the pool of competing strategies in the market. To this end we consider a range of adaptive strategies to assess their performance against the λ^* investment strategy. These competing strategies are time-invariant because they process observations on prices and dividends. Precursors to the numerical study present here are Hens and Schenk-Hoppé (2004) and Hens et al. (2002).

The strategies considered in his simulation are defined as follows. First one has the usual suspects: the Kelly strategy $\lambda_k^* = Ed_k(s) = \frac{1}{26}\sum_{s=1}^{26} d_k(s)$; an illusionary diversification strategy $\lambda_k^{\text{Illu}} = 1/K$; the weighted sample mean of the dividend payments $\lambda_{k,t}^{\text{SMean}} \sim \hat{d}_{k,t} := \sum_{\tau=1}^{t} \beta^{t-\tau} d_k(s_\tau)$ with $\beta = 0.95$; and a strategy with behavioral bias in the sense of Kahneman and Tversky $\lambda_k^{\text{CPT}} \sim \sum_{s=1}^{26} h(d_k(s))$, where the function $h(x)$ is defined as in (Tversky and Kahneman, 1992, Eq. (6), p. 309) with both parameters set to 0.65.

Second there are three 'technical trading' strategies representing investors betting on the trend resp. its reversal (contrarian strategy) as well as on the mean reversion of prices. The definition takes into account that short selling is not permitted:

$$\lambda_{k,t+1}^{\text{Trend}} \sim \left[\frac{p_{k,t}}{p_{k,t-1}} - 1\right]^+, \quad \lambda_{k,t+1}^{\text{Contr}} \sim \left[1 - \frac{p_{k,t}}{p_{k,t-1}}\right]^+, \text{ and } \lambda_{k,t+1}^{\text{MRev}} \sim \left[1 - \frac{p_{k,t}}{\hat{p}_{k,t}}\right]^+$$

If any of the right-hand sides is identical to zero for all assets, the strategy is set to $(1/K, ..., 1/K)$. $\hat{p}_{k,t}$ denotes the sum of discounted realized prices with discount parameter set to 0.95. Let the weighted sample mean gross return $\hat{R}eturn_{k,t}$ be defined analogously.

Finally there are four adaptive investment strategies that are based on the solution of more demanding optimization problems. Their initialization uses the annual returns of the observation period 1986-2006. The optimization is under the constraint of no short-selling and subject to the 'minimum required return' constraint

$$\sum_k (\lambda_k \hat{R}eturn_{k,t}) \geq 3/4 \ \max_k \hat{R}eturn_{k,t} \tag{55}$$

A mean-variance maximization investment strategy $\lambda_{t+1}^{\mu-\sigma}$ is defined as the solution to $\min_{\lambda \in \Delta^I} \lambda \hat{C}_t \lambda^{\mathrm{T}}$, where $\hat{C}_t^{k,j}$ is the weighted sample covariance. A growth-optimal investment strategy that maximizes the discounted logarithmic return based on all realized returns. A 'conditional value-at-risk' portfolio optimization as suggested by Rockafellar and Uryasev with the confidence level set to 5%. A mean-absolute deviation investment strategy as proposed by Konno and Yamazaki.

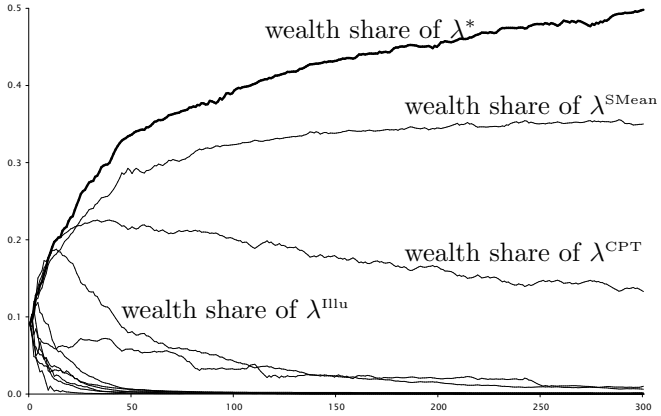


Figure 6: Typical realization of a sample path of the relative wealth of competing investment strategies. All strategies are endowed with the same wealth at time zero.

The simulation result presented in Figure 6 gives a clear message. The constant investment strategy λ^* prevails in the dynamics of wealth shares. The closest competitors are the adaptive strategy λ^{SMean}, which is based on past dividend payments, and the behavioral investment strategy λ^{CPT}, which is a 'distorted' version of λ^*. The poor performance of the chartist strategies as well as the quite sophisticated dynamic strategies is surprising. Another unexpected result is the excellent performance of the illusionary diversification strategy λ^{Illu}. The convergence is considerably slower than in the two-investor case studied above.

The (relative) asset price dynamics corresponding to the sample path of the dividend payments underlying Figure 6 is depicted in Figure 7. Asset prices converge but they are more volatile than one might expect because the sample paths of the wealth dynamics are quite smooth. This observation is explained by the time-variation of the

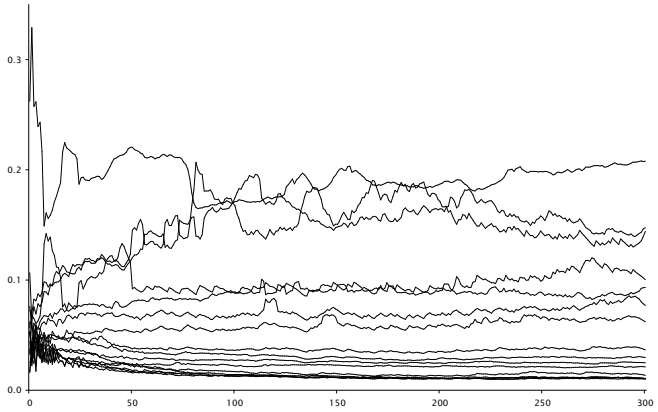


Figure 7: Price dynamics corresponding to Figure 6.

adaptive strategy λ^{SMean} which discounts dividend payments rather than just calculates the sample mean of the relative dividend (which is an unbiased estimator of λ^*).

These findings highlight the need for more simulation studies within this class of dynamic models. Despite extensive numerical work on agent-based models, we see this line of inquiry as a promising area for future research.

5.2 Dynamics of strategies: genetic programming

The dynamics of investment strategies' wealth shares is the main focus in the preceding study. The strategies themselves played a rather static role which is at odds with for instance agent-based models. It is important to recall that strategies in the evolutionary finance models face no restrictions beside the absence of short selling and adaptiveness. Stationarity of strategies is useful in the local stability analysis because the mathematical apparatus is tailored to this framework. The global convergence result for the model with short-lived assets however demonstrate that adapted strategies are a class of investment strategies for which the wealth dynamics can be fully understood.

This leaves open the behavior of models in which the strategies and the dynamics of wealth shares co-evolves. The study of this issue requires an explicit specification of the adaption and innovation of investment behavior and the entry of new strategies. Lensberg and Schenk-Hoppé (2007) pursue a Darwinian approach to the study of the evolution of investment strategies in the model with short-lived assets (which has the advantage of a relative simple computation of the Kelly rule). It adds the other main evolutionary process, reproduction, to the selection mechanism. The framework is that of genetic programming which offers flexibility as well as full control on the data available to investment strategies. The latter is extremely useful in the interpretation of results. In a genetic programming approach the center stage is occupied by the

population which embodies the investment skills of many individual strategies. The investors are simple-minded and unsophisticated in the sense that they follow preprogrammed behavior rules which are the result of mutations and crossovers. While the change in investment behavior is covered by the standard evolutionary finance model, the inflow of new investors requires an extension which however is straightforward.

Two questions are of particular interest. First, is the Kelly rule–as the long-term equilibrium prediction of asset price–valid in a model that imposes much weaker assumptions on the market dynamics? The process of mutation generates an constant inflow of new traders which generates a considerable amount of 'noise' not present in the theoretical studies. Second, will the Kelly rule emerge in the population of traders without strong assumptions on individuals' rationality or learning behavior and despite its absence in the initial population of investors? For instance, Bayesian learning is not an option available to the investors in this genetic programming approach.

Brief description of model Lensberg and Schenk-Hoppé (2007) analyze four cases: complete/incomplete market and i.i.d./Markov states of the world. In each case two different information scenarios are considered as specified below. The total number of investment strategies is limited to 2,000. Each strategy is represented by a computer program that outputs numbers $\tilde{\lambda} : \mathcal{S} \times \{1, \ldots, K\} \to \mathbb{R}$ where the set of inputs $\mathcal{S}$ either contains the information on the current state ($\mathcal{S} = \{1, \ldots, S\}$) or, additionally, the last observed price corresponding to this state ($\mathcal{S} = \{1, \ldots, S\} \times \mathbb{R}^K$). The output is transformed to ensure that budget shares meet our conditions in Section 2.1. A computer program consists of up to 128 lines of instructions, see e.g. (Lensberg and Schenk-Hoppé, 2007, Table I).

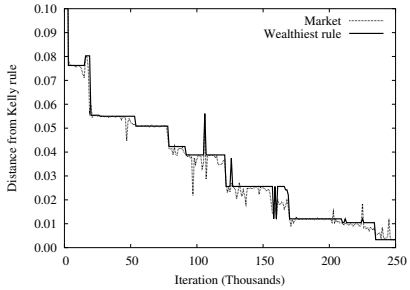
The evolution of the strategies is driven by a tournament selection process. In any one period in time, the following procedure is applied 20 times. Four randomly chosen programs are ranked according to their wealth (tournament). Then the two poorest programs are replaced by the two richest in this sample (reproduction). These two clones have, with some probability, a randomly selected instruction replaced by some random code (mutation). Finally, again with some probability, a randomly selected set of instructions is swapped between (crossover). Completely new behavior is introduced by adding a random draw that decides whether the programs are filled with random instructions (noise). All the investors in a tournament retain their wealth, except if it is zero which entitles the respective investor to an endowment of 1% of the total wealth. The simulations reveal a substantial number of investment strategies without wealth.

Consider a market consisting of Arrow-type securities in which the state of the world is Markovian and programs only have access to the current state of the world (Lensberg and Schenk-Hoppé, 2007, Section 3.2):

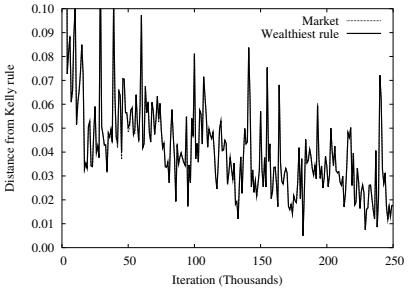
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad (\pi(s|\hat{s}))_{s,\hat{s}=1,2,3} = \begin{pmatrix} .7 & .2 & .1 \\ .1 & .7 & .2 \\ .2 & .1 & .7 \end{pmatrix}$$

If the only information given to the programs representing the investment strategies consists of the current state of the world, strategies' behavior is mainly a 'bet their beliefs' style. For price-dependent strategies a more realistic scenario is obtained when

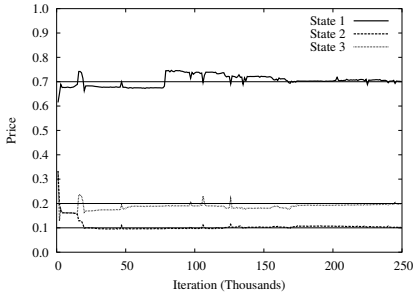
providing programs with additional information about the last observed price system corresponding to the current state of the world. This in particular enables the purchase of an approximate market portfolio. The latter is simply achieved by outputting these prices as budget shares.



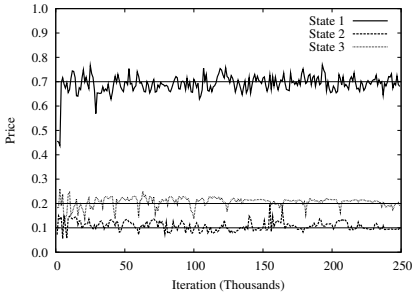
(a) Distance between market prices resp. wealthiest strategy and Kelly rule



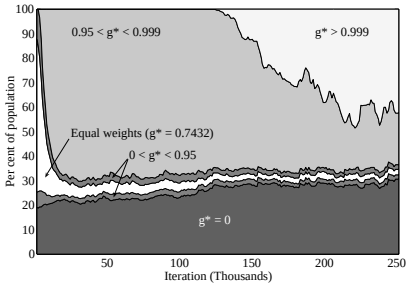
(b) Distance between market prices resp. wealthiest strategy and Kelly rule (nearly identical)



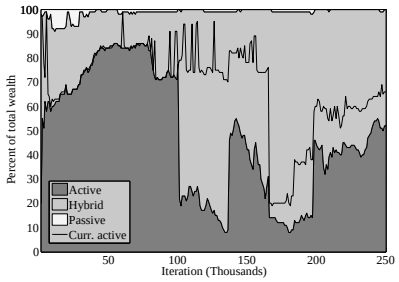
(c) Relative price of Asset 1 for all three states; horizontal lines indicate the Kelly benchmark



(d) Relative price of Asset 1 for all three states; horizontal lines indicate the Kelly benchmark



(e) Distribution of growth rates in the population relative to the Kelly rule



(f) Distribution of growth rates in the population relative to the Kelly rule

Figure 8: State-dependent strategies (left panels) and price-dependent strategies (right panels).

Results The results for these two different specifications of the information set are summarized in Figure 8. The two top panels illustrate the convergence to the Kelly rule of both market and wealthiest investment strategy. The two middle panels give the times series of asset prices. The bottom panels provides some insight in the distribution of 'investment skills' within the population. The graphs show how close investment strategies are to the Kelly rule in the population of traders. g^* is the exponential of the expected logarithmic growth rate at λ^*-prices.

For state-dependent strategies the distinctive features are almost monotone convergence, leapfrogging of the distance, and the observation that the market leads relative to the wealthiest investment strategy (which is determined in each period). The population quickly moves towards investment strategies that are quite close to the Kelly rule. The emergence of nearly perfect matches however takes comparatively long. The effect of the noise, i.e. the continuing introduction of randomly generated strategies, is clearly documented by the persistence of a large number of poorly performing strategies (bottom left panel of Figure 8).

When strategies have access to the last observed price system which corresponds to the current state of the world, convergence still occurs but the pattern exhibits much more volatility. The long-term outcome is again the Kelly rule for both wealthiest investment strategy and market prices. An analysis of wealthy investment strategies reveals a new type of behavior: almost all successful strategies use the (proxy) market portfolio. The fraction of wealth invested in the market portfolio as well as the closeness of the budget shares to the market portfolio vary with the difference between market prices and the Kelly rule as well as the specific prices. Figure 8 (d) illustrates the composition of the pool of investment strategies. Strategies can be classified by their deviation from the market portfolio. Active – always deviate; hybrid – deviate only for some asset prices; and passive – never deviate. For a particular run, the number of hybrid strategies that are currently deviating from the market portfolio varies.

Anatomy of successful strategies Let us "dissect" the wealthiest strategy at the end of the simulation period, λ^{LW}. The behavior of this strategy in state $s = 3$ is summarized in Figure 9. The triangle is the set of all possible price vectors (q_1, q_2, q_3). The darker an area, the larger the distance of the budget shares to the market portfolio. White areas correspond to prices at which the strategy's portfolio is identical to the market portfolio.

The strategy λ^{LW} is hybrid and has a trigger that switches from active to passive investment. If the price of asset 1 is low, the market portfolio is played. This also happens if the price of asset 2 coincides with the Kelly rule. For all other prices, the strategy deviates from the market portfolio. Its functional form in the active mode is given by

$$\lambda^{LW}(q) = (1 - \lambda_2^* - \epsilon(q_2)) \left(\frac{q_1}{q_1 + q_3}, \frac{\lambda_2^* + \epsilon(q_2)}{1 - \lambda_2^* - \epsilon(q_2)}, \frac{q_3}{q_1 + q_3} \right)$$

with a convex function $\epsilon(p_2)$ with values $\epsilon(0) \approx 0.008$, $\epsilon(\lambda_2^*) \approx 0.0$ and $\epsilon(1) \approx -0.016$. Strategy λ^{LW} makes bets on a reversal of the price of asset 2 to the Kelly rule. The remaining wealth is invested in a market portfolio consisting only of the two assets 1 and 3. The behavior leads to a convergence of the price of asset 2 to the Kelly

benchmark, if λ^{LW} becomes wealthy. Other strategies are specialized in very similar fashions, but for different assets and states.

The optimality properties of investing part of the wealth so as to reduce the risk associated to the volatility of the portfolio return (the "fractional Kelly rule") are e.g. discussed in MacLean et al. (1992).

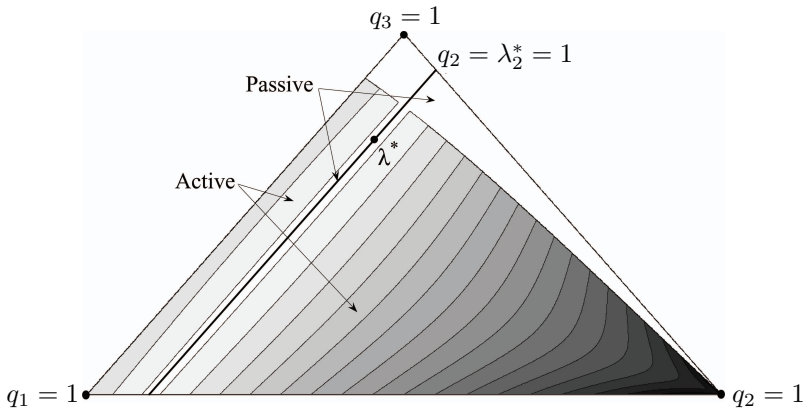


Figure 9: Price-dependent strategies. Contour plot of $\delta(q) := \|\lambda^{LW}(q) - q\|$ for relative prices $q = (q_1, q_2, q_3)$ in the unit simplex. $\delta(q)$ is the Euclidean distance between the market portfolio p and the portfolio weights of strategy λ^{LW} in state 3. Darker areas represent larger values of $\delta(q)$, and white areas represent those prices for which λ^{LW} is in perfect agreement with the market portfolio, i.e., $\delta(q) = 0$. Kelly prices and Kelly portfolio weights are given by $\lambda^* = (0.2, 0.1, 0.7)$.

Conclusion The simulation results of the genetic programming approach to the dynamics and mutation of investment strategy confirms the pivotal role of the Kelly rule. The long-term outcome of the market dynamics is fully described by the Kelly rule—this outcome is also robust against noise. The numerical study also highlights the importance of the market portfolio in this class of models. The market portfolio (even if it is only a proxy) provides insurance against severe losses. In the present genetic programming framework too high volatility of returns is punished by the tournament process that annihilates poor strategies. Surprisingly perhaps, the typical trader types that are assumed to populate the market in noise-trader or agent-based models do not enter the stage.

5.3 Empirical tests of evolutionary asset pricing

In this section two of the theoretical evolutionary finance results are tested empirically. First, the prediction of asset prices which is derived from the long-term dynamics of the market: the Kelly rule as a benchmark for the (relative) fundamental valuation of assets. Second, the market dynamics which, in the presence of a Kelly investor, describes the convergence of relative asset prices to the Kelly benchmark. The latter

highlights the strength of evolutionary finance models which overcomes the shortcomings of equilibrium models in which these convergence dynamics are mainly an exercise in semantics because this dynamics simply is not modeled.

Empirical support for this dynamic approach and its predictions has interesting implications. In this case evolutionary finance can shed light on the issue of excess returns in financial markets which are a hot topic ever since these markets came in existence. A prominent example are excess returns from value investment, i.e. bets on the reversal of prices to some fundamental value such as price-to-book ratio or dividend yield. Graham and Dodd (1934), who were the main proponents of this investment advice, conjectured that excess returns from value investment originate from a tendency of markets to converge towards fundamental values. A simple approach to profit from this price dynamics is to go short in overvalued assets (whose price falls) and long in undervalued assets (whose price increases). This line of thought is explored in the empirical study of the value premium puzzle by Hens et al. (2008).

The empirical test employs the evolutionary finance model with long-lived assets, Section 4. Each time period is interpreted as one year, and the asset payoffs are given by the vector with each firm's total dividend payment in that year. The data sample consists of all 16 firms that were listed in the Dow Jones Industrial Average (DJIA) index during the time period 1981-2006. The data are taken from CRSP.

Hypothesis 1 (relative asset prices are determined by λ^*)

Our results state that the relative market capitalization of an asset is (asymptotically) given by the expected value of its discounted relative payoffs. The relative market capitalization of a firm (denoted by $q_{k,t}$) is simply calculated from the stock prices and the number of shares issued for all firms in the sample. How to determine the relative fundamental value however is less straightforward and, obviously, leaves the econometrician with many options. We take the current relative dividend of each firm (denoted by $R_{k,t}$, k the index of the firm) as a proxy for the relative fundamental value (the Kelly rule $\lambda^*_{k,t}$).

Our (joint) hypothesis is that in the linear cross-sectional regression

$$q_{k,t} = a_0(t)R_{k,t} + a_1(t) + \epsilon_t, \quad k = 1, ..., 16 \tag{56}$$

$a_0(t) > 0$ and $a_1(t) = 0$ for $t = 1981, ..., 2006$. If this relation holds then, in each year, the relative market capitalization of a firm depends linearly on its current relative dividend payment.

Hypothesis 2 (convergence of relative asset prices to λ^*)

The convergence of prices to the Kelly prices λ^* is a consequence of the market dynamics. If the previous hypothesis has sufficient empirical support, one can study the dynamics of small deviations from the benchmark $\lambda^*_{k,t}$. This empirical benchmark will be defined as the valuation derived in the study of hypothesis 1. Suppose there is one $\lambda^*_{k,t}$ investor and a mutant investment strategy $\mu_{k,t}$ representing all the other investors in the market. Exponentially fast convergence of the Kelly investor's wealth share $r^*_t \to 1$ can be expressed as $[1 - r^*_{t+1}] = \alpha_t[1 - r^*_t]$ with some variable α_t, $0 < \alpha_t < 1$. The mean value of this parameter is determined by the exponential of the logarithmic growth rate $g_{\lambda^*}(\mu)$ as defined in (45). Since

$$q_{k,t} = \lambda^*_{k,t} r^*_t + \mu_{k,t}(1 - r^*_t)$$

one obtains (after some elementary calculations) the relation

$$[\lambda^*_{k,t+1} - q_{k,t+1}] = \alpha_t \frac{\lambda^*_{k,t+1} - \mu_{k,t+1}}{\lambda^*_{k,t} - \mu_{k,t}} [\lambda^*_{k,t} - q_{k,t}] \tag{57}$$

Our hypothesis is formalized as follows. Between any two consecutive years, t and $t+1$, $t = 1981, ..., 2005$, the linear regression

$$[\lambda^*_{k,t+1} - q_{k,t+1}] = a(t) [\lambda^*_{k,t} - q_{k,t}] + \epsilon_t, \ k = 1, ..., K \tag{58}$$

has a least-squares estimator $0 < a(t) < 1$ and ϵ_t is a noise term with mean zero.

The empirical results are summarized in Table 1. The hypothesis 1 on the relevance of the Kelly rule as a pricing benchmark (for the relative valuation of firms) is strongly supported. In every year of the sample the coefficient $a_0(t)$ is significantly positive. In addition, the coefficient $a_1(t)$ is not significantly different from zero. The adjusted R^2 values indicate that a considerable amount of the variation in the data is explained by the model, see Table 1(a). The hypothesis 2 on the convergence of relative market capitalization towards the benchmark is supported by the empirical findings. Most of the coefficients $a(t)$, see Table 1(b), are between zero and one. This finding is statistically significant on the 1% significance level. The adjusted R^2 values are quite high which indicates that the model has strong explanatory power. In seven of the 25 years of observation, the coefficient $a(t)$ is larger than one (which implies divergence from the benchmark from the current to the next year). The hypothesis that the coefficient is less than one however cannot be rejected at the 1% level. In summary, both hypotheses are strongly supported by the empirical results.

The empirical analysis presented here is certainly not more than a preliminary assessment of the potential of evolutionary finance in explaining asset prices and their dynamics. This topic merits additional (and more thorough) inquiry.

6 Continuous-time evolutionary finance

This section presents recent progress in advancing the evolutionary finance approach in the direction of continuous-time financial mathematics. The development of such an approach is of interest because it builds on the workhorse model of financial mathematics and it allows for different time scales for trading and changes in dividend payments. The main conceptual innovation is the introduction of the market interaction of heterogenous investors with self-financing investment strategies—and, thus, endogenous prices—in this framework. The model accommodates e.g. different time scales for the frequency/intensity of trades and dividend payments. This offers an alternative approach to the study of the price impact of large trades. The mathematical theory used to formulate the continuous-time evolutionary finance model is that of random dynamical systems with continuous time, Arnold (1998).

The analysis focuses, as in the discrete-time model, on the asymptotic dynamics of the wealth distribution and asset prices. The derivation of convergence results however requires the application of very different mathematical techniques. For simplicity of

presentation only the continuous case (without jumps) is considered here. Details and proofs can be found in Palczewski and Schenk-Hoppé (2008).

There are K assets (stocks), each in constant supply of one. Denote the price process, which will be described later, by $S(t) = (S_1(t), \ldots, S_K(t))$ and the cumulative dividend payment by $D(t) = (D_1(t), \ldots, D_K(t))$, $t \geq 0$. There are I investors. The portfolio of investor i is denoted by $\theta^i(t) = (\theta^i_1(t), \ldots, \theta^i_K(t))$, and his cumulative consumption process is given by $C^i(t)$. For a self-financing portfolio-consumption process $(\theta^i(t), C^i(t))$, the dynamics of investor i's wealth $V^i(t) = \sum_{k=1}^K \theta^i_k(t) S_k(t)$ is given by

$$dV^i(t) = \sum_{k=1}^K \theta^i_k(t)\big(dS_k(t) + dD_k(t)\big) - dC^i(t) \tag{59}$$

Self-financing means that changes in value can be attributed either to changes in asset prices, dividend income or consumption expenditure. An investor's portfolio can be written as $\theta^i_k(t) = \lambda^i_k(t) V^i(t)/S_k(t)$ with a real-valued process $\lambda^i(t) = (\lambda^i_1(t), ..., \lambda^i_K(t))$ as investment strategy. Since assets are in net supply of one, market-clearing implies

$$S_k(t) = \lambda^1_k(t)\, V^1(t) + \ldots + \lambda^I_k(t)\, V^I(t) = \langle \lambda_k(t), V(t) \rangle \tag{60}$$

i.e. every asset's market value is equal to the aggregate investment in that asset, cf. (9). This defines a market-clearing price for given investment strategies and wealth distribution. One obtains

$$dV^i(t) = \sum_{k=1}^K \frac{\lambda^i_k(t) V^i(t)}{\langle \lambda_k(t), V(t) \rangle} \big(d\langle \lambda_k(t), V(t) \rangle + dD_k(t)\big) - dC^i(t) \tag{61}$$

for all $i = 1, ..., I$.

Suppose there are $I = 2$ investors with time-invariant investment strategies, i.e. $\lambda^i_k(t) \equiv \lambda^i_k$, and consumption process $dC^i(t) = c\, V^i(t) dt$. The constant $c > 0$ is the consumption rate; it is assumed to be the same for all investment strategies. Assume the cumulative dividend process of each asset can be written using an intensity process, i.e. $dD_k(t) = \delta_k(t) dt$ with $\delta_k(t) \geq 0$, and that $\bar{\delta}(t) = \sum_{k=1}^K \delta_k(t) > 0$. Then the dynamics of the relative wealth of the investment strategy λ^1, $w^1(t) = V^1(t)/[V^1(t) + V^2(t)]$, is given by

$$dw^1(t) = c\, w^1(t)\, \frac{\sum_{k=1}^K \frac{\lambda^1_k}{[\lambda^1_k - \lambda^2_k] w^1(t) + \lambda^2_k} \rho_k - 1}{\sum_{k=1}^K \frac{\lambda^1_k \lambda^2_k}{[\lambda^1_k - \lambda^2_k] w^1(t) + \lambda^2_k}}\, dt \tag{62}$$

with $\rho_k(t) = \delta_k(t)/[\delta_1(t) + \delta_2(t)]$, $k = 1, 2$, the relative dividend intensity. The dynamics (62) is well-defined if $\rho_1(t)$ (and thus $\rho_2(t)$) is locally integrable, see (Palczewski and Schenk-Hoppé, 2008, Lemma 2). (The relative wealth of the other investment strategy is given by $w^2(t) = 1 - w^1(t)$.)

Suppose further there are $K = 2$ assets. Then (62) can be factorized as (provided $\lambda^1 \neq \lambda^2$)

$$dw^1(t) = c \frac{-w^1(t)\big(1 - w^1(t)\big)\big((\lambda^2_1 - \lambda^1_1)^2 w^1(t) + (\lambda^2_1 - \lambda^1_1)(\rho_1(t) - \lambda^2_1)\big)}{\big(\lambda^1_1 \lambda^1_2 - \lambda^2_1 \lambda^2_2\big) w^1(t) + \lambda^2_1 \lambda^2_2}\, dt \tag{63}$$

We finally assume that

$$\lambda_1^* = \lim_{t\to\infty} \frac{1}{t} \int_0^t \rho_1(u)\, du \tag{64}$$

is well-defined ($\lambda_2^* = 1 - \lambda_1^*$). Then one has (Palczewski and Schenk-Hoppé, 2008, Theorems 1 and 2)

Theorem 6.1. *Let $\lambda^2 = \lambda^*$, and assume that $\lambda^1 \neq \lambda^2$. Fix any initial value $w_1(0) \in (0,1)$.*

(a) $\lim_{t\to\infty} \frac{1}{t} \int_0^t w_1(u)\, du = 0$.

(b) Suppose there is a real number γ such that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}_{(-\infty,\gamma)}\Big(\operatorname{sgn}(\lambda_1^* - \lambda_1^1) \int_s^t (\rho_1(u) - \lambda_1^*)\, du\Big)\, ds > 0 \tag{65}$$

Then the relative wealth of investor 1 converges to 0 (while that of investor 2 converges to 1), i.e.

$$\lim_{t\to\infty} w_1(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} w_2(t) = 1$$

Theorem 6.1 states that the wealth dynamics selects the investor who divides wealth according to the time-average of the relative dividend intensity. This finding is in line with our previous analysis. Part (a) asserts convergence in the Cesàro sense. Counter examples show that the stronger convergence in part (b) cannot be obtained without additional conditions.

It is of interest to note that the speed of convergence of $w^1(t) \to 0$ in Theorem 6.1 is not exponentially fast. This is at odds with the corresponding models in discrete time, see Sections 3.2 and 4.1. Suppose $\rho(t)$ is a stationary ergodic process with the stationary measure μ. Then $\lambda^* = E^\mu \rho$. The linearization at the steady state $w^1(t) = 0$ gives the variational equation

$$dv(t) = c\, \frac{\lambda_1^1 - \lambda_1^2}{\lambda_1^2 \lambda_2^2} \big(\rho_1(t) - \lambda_1^2\big)\, v(t)\, dt$$

which shows that the exponential growth rate of $v(t)$ is equal to

$$c\, \frac{\lambda_1^1 - \lambda_1^2}{\lambda_1^2 \lambda_2^2} \big(E^\mu \rho_1 - \lambda_1^2\big)$$

If $\lambda^2 = \lambda^*$, the exponential growth rate is equal to zero for *every* investment strategy λ^1. For any time-invariant investment strategy $\lambda^2 \neq \lambda^*$, however, there is an investment strategy λ^1 such that the growth rate is strictly positive, i.e. $v(t)$ diverges from 0 exponentially fast. If $\lambda_1^2 < \lambda_1^*$, take any $\lambda_1^1 \in (\lambda_1^2, 1)$; otherwise take $\lambda_1^1 \in (0, \lambda_1^2)$.

The condition (65) is satisfied for a large class of processes. Assume for instance that the dividend intensity process $\rho_1(t)$ is a positively recurrent Markov process on a countable subset of $[0,1]$. Denote the unique invariant probability measure by μ. Then, in (64), $\lambda_1^* = \bar\rho = E^\mu \rho_1(0)$ is well-defined. Let P^μ denote the probability

measure under which the distribution of $\rho_1(0)$ is given by μ. Theorem 3 in Palczewski and Schenk-Hoppé (2008) ensures that if

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \int_0^t P^\mu\left(\frac{1}{s}\int_0^s \rho_1(u)du < \bar{\rho}\right) ds > 0 \\ \lim_{t\to\infty} \frac{1}{t} \int_0^t P^\mu\left(\frac{1}{s}\int_0^s \rho_1(u)du > \bar{\rho}\right) ds > 0 \end{aligned} \tag{66}$$

then (65) holds for $\gamma = 0$. Therefore, $\lim_{t\to\infty} w^1(t) = 0$.

For instance, if the process $\rho_1(t)$ has initial distribution μ and is symmetric around its expected value (and takes on at least two different values), then (66) is satisfied. It is also sufficient if the first return time is square integrable for at least one element of the state space E, see (Palczewski and Schenk-Hoppé, 2008, Sect. 4).

An interesting topic for future research on evolutionary finance models in continuous time is the study of (61) with adapted, time-variant investment strategies (possibly more investors and more assets). Another line of inquiry is concerned with the corresponding diffusion-type model which requires the use of stochastic analysis.

7 Conclusion

This chapter surveyed current research on and applications of evolutionary finance which is inspired by Darwinian ideas and random dynamical systems theory. This approach studies the market interaction of investment strategies—and the wealth dynamics it entails—in financial markets. We were particularly interested in the long-term dynamics of the wealth distribution with the goal of identifying surviving investment strategies and the corresponding asset price system. The emphasis in this survey was on the motivation and the heuristic justification of the results, technical details were avoided as far as possible. In contrast to the current standard paradigm in economic modeling, we pursued an approach that is based on random dynamical systems. Equilibrium holds only in the short-term, which reflects the model of investment behavior explored in our evolutionary finance approach.

The motivation was derived in the context of a model of betting markets which goes back to Kelly's 1956 paper. The modeling approach and its main components and assumptions were explained in detail, Section 2. The main part of the chapter was devoted to the two main modeling frameworks: models with short-lived assets (bets) and those with long-lived assets (stocks). In each case, the analysis moved from a (relatively) simple to more demanding settings in which more advanced mathematical techniques were required and the proofs became more involved. In the simplest case considered here, investment strategies are constant vectors and asset payoffs are driven by an i.i.d. process. In the most advanced case, the first were adapted processes while the latter were governed by Markov processes.

Models with short-lived assets were covered in Section 3. Both local and global dynamics were studied, and some numerical simulations were presented. This model is a generalization of the betting market setting considered by Kelly. Surprisingly, results do not depend whether the asset market is complete or incomplete (more states than

assets). An evolutionary stock market model (with long-lived assets) was the subject of Section 4. In this class of models investors are exposed to capital gains/losses which are induced by the price dynamics of the assets. This feature has a considerable impact on the wealth dynamics and its quantitative study. All results obtained in this framework were presented and explained in detail.

Applications of evolutionary finance models for both short- and long-lived assets were presented in Section 5 which comprises simulation and empirical studies. The numerical studies explored the dynamics beyond the setting in which the analytical results were obtained. We simulated the wealth and asset price dynamics in scenarios with different types of investment strategies—and in the absence of the generalized Kelly rule. The evolution (or mutation) of strategies, rather than just the wealth dynamics of pre-specified investment strategies, was numerically analyzed by combining the standard evolutionary finance model with genetic programming and tournament selection. The section closed with the presentation of recent empirical results on the explanatory power of evolutionary finance in real markets.

Continuous-time evolutionary finance models, presented in Section 6, are the latest development in this field. This approach can be seen as a generalization of the workhorse model of continuous-time financial mathematics. We introduced endogenous prices via short-term market clearing in this model using the same ideas as in discrete time. One advantage of this model is the flexibility to have different frequency of trade and changes of dividend payments.

Several proposals for future research topics within evolutionary finance were made throughout this chapter. One main task will be to study the game-theoretic perspective of evolutionary finance which is not satisfactorily explored yet. Among the challenging (as well as most rewarding) subjects we highlighted the need for additional empirical studies and the further development of continuous-time evolutionary finance.

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(a) Coefficients, probabilities and R^2 of the regression (56) testing the asset pricing hypothesis 1.

Year t	$a_0(t)$	P-value	$a_1(t)$	P-value	R^2 adj.
1981	0.550	0.000	0.028	0.011	0.671
1982	0.584	0.000	0.026	0.001	0.835
1983	0.613	0.000	0.024	0.005	0.799
1984	0.643	0.000	0.022	0.036	0.710
1985	0.622	0.000	0.024	0.027	0.713
1986	0.609	0.000	0.025	0.058	0.577
1987	0.474	0.000	0.033	0.009	0.485
1988	0.515	0.000	0.030	0.012	0.516
1989	0.549	0.000	0.028	0.013	0.561
1990	0.243	0.030	0.047	0.001	0.145
1991	0.280	0.030	0.045	0.002	0.146
1992	0.328	0.031	0.042	0.006	0.141
1993	0.508	0.003	0.031	0.026	0.315
1994	0.496	0.001	0.032	0.014	0.364
1995	0.575	0.000	0.027	0.020	0.474
1996	0.606	0.000	0.025	0.039	0.442
1997	0.596	0.000	0.025	0.041	0.421
1998	0.743	0.000	0.016	0.125	0.556
1999	0.795	0.000	0.013	0.212	0.518
2000	0.707	0.001	0.018	0.171	0.376
2001	0.891	0.000	0.007	0.154	0.917
2002	0.776	0.000	0.014	0.077	0.816
2003	0.692	0.000	0.019	0.009	0.832
2004	0.674	0.000	0.020	0.004	0.842
2005	0.803	0.000	0.012	0.061	0.873
2006	0.822	0.000	0.011	0.078	0.877

(b) Coefficient, probabilities and adjusted R^2 of the regression (58) testing the convergence hypothesis 2.

Year t	$a(t)$	P-value	R^2 adj.
1981	0.529	0.000	0.677
1982	0.865	0.000	0.524
1983	1.185	0.000	0.833
1984	0.973	0.000	0.943
1985	1.233	0.000	0.970
1986	0.814	0.000	0.852
1987	0.871	0.000	0.875
1988	0.907	0.000	0.967
1989	1.031	0.000	0.757
1990	1.049	0.000	0.945
1991	0.862	0.000	0.752
1992	0.716	0.000	0.673
1993	0.784	0.000	0.731
1994	0.815	0.000	0.886
1995	0.982	0.000	0.903
1996	1.002	0.000	0.885
1997	0.847	0.000	0.776
1998	1.076	0.000	0.901
1999	1.080	0.000	0.733
2000	0.115	0.113	0.089
2001	0.875	0.003	0.329

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