MEASURABLE SELECTIONS AND REGULAR EXPECTATIONS

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Outline of the theme

Measurable selection theorems state that from a set depending measurably on a parameter, one can select a point depending measurably on the parameter. There is a whole variety of results of this kind obtained under different assumptions. This mathematical area lies at the interface of measure theory, general topology and set theory, dealing sometimes with the most abstract mathematical objects and techniques. In this connection it might be interesting to note that the most widely used measurable selection theorem is due to Nobel Laureate in Economics Robert Aumann [1]. It was developed as a tool in Aumann's seminal work on economies with atomless measure spaces of agents.

[1] Aumann R. J.: Measurable utility and the measurable choice theorems. La Decision, 2 (Actes Coll. Int. CNRS, Aix-en-Provence, 1967), 15-26.

Classical measurable selection theorems deal with complete separable metric spaces with Borel sigma-algebras. Papers [2,3] extend them to various classes of general topological spaces endowed with Baire and Borel measurable structures.

[2] Measurable selection theorems and stochastic control models in general topological spaces, 1986, Mat. Sbornik (Math. USSR Sbornik), v. 131, n. 1, 27-39. PDF
[3] Measurable images of compact spaces and selectors of analytic sets, 1988, Doklady AN SSSR (Soviet Math. Dokl.), v. 299, n. 3, 538-541. PDF

Regular conditional expectations. In Kolmogorov's measure-theoretic framework underlying modern probability theory, the conditional expectation of a random variable is defined, strictly speaking, not as some particular random variable, but as a class of random variables coinciding almost surely. Sometimes, especially if we are dealing with a whole family of random variables, we need to select a representative in this class in a certain "good" (consistent) manner. A standard way of doing it is to use the so-called regular conditional probabilities – the notion which can be found in any advanced textbook on Probability. However, regular conditional probabilities do not exist always, only under some assumptions on the probability space. To overcome this difficulty paper [4] proposed an alternative concept, the regular conditional expectation (RCE) of a random variable depending on a parameter. RCEs exist in the most general situations, while their uniqueness turns out to be equivalent to the validity of the measurable selection theorem for the space under consideration. Paper [5] extended regular conditional expectations to set-valued random variables, and papers [6,7] examined their properties in various settings.

[4] Evstigneev I. V.: Measurable selection and dynamic programming, Math. Oper. Res., 1 (1976) 52-55. PDF

[5] Regular conditional expectations of correspondences, Teor. Ver. i Primen.(Theory of Probab. and Appl.) 21 (1976) 333-347 (with E.B. Dynkin). PDF
[6] Regular conditional expectations of random variables depending on parameters, 1986, Teor. Ver. i Primen. (Theory of Probab. and Appl.) 31 (1986) 586-589. PDF
[7] Regular conditional expectations and the continuum hypothesis, in: The Dynkin Festschrift, M.I. Freidlin, ed., Birkhäuser, Boston, 1994, 85-93. DOI