Stochastic Nonlinear Perron-Frobenius Theorem and von Neumann-Gale Dynamical Systems

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Von Neumann-Gale dynamical systems are defined in terms of multivalued operators possessing properties of convexity and homogeneity. These operators assign to each element of a given cone a convex subset of the cone describing possible one-step transitions from one state of the system to another. Key results on von Neumann-Gale dynamical systems may be regarded as multivalued nonlinear versions of the Perron-Frobenius theorem.

The classical, deterministic theory of such dynamics was originally aimed at the modeling of economic growth (von Neumann 1937 and Gale 1956). First attempts to build a stochastic generalization of this theory were undertaken in the 1970s by Dynkin, Radner and their research groups. However, the initial attack on the problem left many questions unanswered. Substantial progress was made only in the late 1990s, and final solutions to the main open problems were obtained only in the last 5-6 years. The talk will review this theory and outline its recently discovered applications in finance.

Multivalued dynamical systems

Given:

Set X_t (*state space* at time t), $t = 0, 1, 2, \ldots$;

multivalued mapping

$$x \mapsto A_t(x), \quad x \in X_{t-1}, \quad A_t(x) \subseteq X_t$$

(*transition mapping*).

Paths, or *trajectories*: sequences $x_0, x_1, \ldots$ such that

$$x_t \in A_t(x_{t-1}).$$

Von Neumann-Gale dynamical systems

X_t are **cones** in linear spaces;

for each t, the graph of the transition mapping $A_t(\bullet)$,

$$Z_t = \{(x,y) \in X_{t-1} \times X_t \,:\, y \in A_t(x)\},$$

is a **cone**.

Equivalent description in terms of transition cones:

Given: *transition cones* Z_t; *paths* are sequences $x_0, x_1, \ldots$ such that

$$(x_{t-1}, x_t) \in Z_t.$$

Autonomous systems

X_t and $A_t(\bullet)$ (or Z_t) do not depend on t.

Example: **von Neumann** (**1937**) **model of economic growth**

$$X_t = R_+^n,$$

$$Z_t \subseteq R_+^n \times R_+^n \text{ polyhedral cones.}$$

States $x = (x^1, \ldots, x^n) \geq 0$ are *commodity vectors*. The process of economic growth: dynamics of x_t in time. Feasible growth paths $x_0, x_1, \ldots$:

$$(x_{t-1}, x_t) \in Z_t \,.$$

Elements $(x, y) \in Z_t$ are feasible *input-output pairs*, or *technological processes* (for the time period $t-1, t$). Z_t are termed *technology sets*.

The cone Z_t is *polyhedral*: there is a finite set of *basic* technological processes

$$(x^1_{t-1}, y^1_t), \ldots, (x^m_{t-1}, y^m_t)$$

and

$$(x, y) \in Z_t \iff (x, y) = \sum_{j=1}^{m} d^j (x^j_{t-1}, y^j_t),$$

where

$$d^1 \geq 0, \ldots, d^m \geq 0$$

are *intensities* of operating the technological processes

$$(x^1_{t-1}, y^1_t), \ldots, (x^m_{t-1}, y^m_t).$$

Gale (**1956**): general, not polyhedral, cones.

The main notions related to v. N.-G. systems: deterministic case

Assume the system is autonomous: $Z_t = Z \subseteq R_+^n \times R_+^n$.

Path: sequence (x_t) such that $(x_{t-1}, x_t) \in Z$.

Dual path: sequence (p_t) such that

$$p_t y \leq p_{t-1} x \text{ for all } (x, y) \in Z.$$

This implies: if (p_t) is a dual path, then for any path (x_t):

$$p_0 x_0 \geq p_1 x_1 \geq p_2 x_2 \geq \ldots$$

A dual path (p_t) **supports** a path (x_t) if

$$p_t x_t = 1.$$

Balanced path: $x_t = \lambda^t x$.

Von Neumann path: that balanced path for which λ is the *greatest*.

Balanced dual path: $p_t = \mu^{-t} p,\ \mu > 0$.

Von Neumann equilibrium: triplet (λ, x, p) such that $\lambda^t x$ is a balanced path and $\lambda^{-t} p$ is a dual path supporting it.

Under general assumptions, (λ, x, p) *is an equilibrium if and only if*

$$(x, \lambda x) \in Z,$$

$$\frac{py}{\lambda} \leq px \text{ for all } (x, y) \in Z,$$

$$px = 1.$$

Economic meaning: λ growth factor of the economy; $1/\lambda$ discount factor; p equilibrium prices.

Von Neumann (1937): existence of equilibrium for a polyhedral Z.

Relation to the Perron-Frobenius theorem

Let

$$Z = \{(x,y) \ : \ y = Ax\},$$

where A is a non-negative matrix; $A^m > 0$ for some m. Then (λ, x, p) *is a von Neumann equilibrium if and only if*

$$\lambda x = Ax, \ \lambda^{-1} p = pA, \ xp = 1.$$

i.e., λ and x are the P-F eigenvalue and eigenvector of A and λ^{-1} and p are the P-F eigenvalue and eigenvector of the conjugate of A.

Von Neumann, **J**.: Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes. In: *Ergebnisse eines Mathematischen Kolloquiums*, No. 8, 1935-1936, Franz-Deuticke, 1937, pp. 73–83.

Translated into English: A model of general economic equilibrium, *Review of Economic Studies* 13 (1945-1946), 1–9.]

Gale, **D**.: A closed linear model of production, in: *Linear Inequalities and Related Systems* (H.W. Kuhn and A.W. Tucker, eds.), pp. 285–303, Princeton University Press, Princeton, 1956.

Rockafellar, **R.T**.: Monotone Processes of Convex and Concave Type, *Memoirs of the American Mathematical Society*, Volume 77, AMS, Providence, 1967.

Stochastic von Neumann–Gale dynamical systems

Pioneering work of Eugene Dynkin, Roy Radner and their research groups in the 1970s.

Given:

Probability space $(\Omega, \mathbf{F}, P)$;

filtration $\mathbf{F}_0 \subseteq \mathbf{F}_1 \subseteq \ldots \subseteq \mathbf{F}_t \subseteq \ldots \subseteq \mathbf{F}$;

set-valued *transition mappings*

$$(\omega, a) \mapsto A_t(\omega, a) \subseteq R^n_+$$

assigning to each $\omega \in \Omega$ and $a \in R^n_+$ a set $A_t(\omega, a) \subseteq R^n_+$ such that

(i) for each ω, the graph

$$Z_t(\omega) := \{(a, b) : b \in A_t(\omega, a)\}$$

of the mapping $A_t(\omega, \cdot)$ is a closed convex cone (*transition cone*);

(ii) the set-valued mapping $Z_t(\omega)$ is $\mathbf{F}_t$-measurable.

Paths (trajectories) $x_0(\omega), x_1(\omega), \ldots$

$$x_t(\omega) \in A_t(\omega, x_{t-1}(\omega)) \text{ (a.s.)},$$

or, equivalently,

$$(x_{t-1}(\omega), x_t(\omega)) \in Z_t(\omega) \text{ (a.s.)}$$

and

$$x_t \text{ is } \mathbf{F}_t\text{-measurable.}$$

Let $\mathbf{X}_t$ denote the space of $\mathbf{F}_t$-measurable random vectors and put

$$\mathbf{Z}_t := \{(x, y) \in \mathbf{X}_{t-1} \times \mathbf{X}_t : (x(\omega), y(\omega)) \in Z_t(\omega) \text{ a.s.}\}.$$

Thus, **a path is a sequence** $x_0, x_1, x_2, \ldots$ **such that**

$$(x_{t-1}, x_t) \in \mathbf{Z}_t.$$

Applications of von Neumann-Gale systems in finance: a dynamic securities market model

n **assets**;

a (contingent) **portfolio** of assets

$$x_t(\omega) = (x_t^1(\omega), \ldots, x_t^n(\omega)), \; x_t \in \mathbf{X}_t.$$

trading strategy: a sequence of portfolios

$$x_0, x_1, x_2, \ldots;$$

self-financing trading strategy:

$$(x_{t-1}, x_t) \in \mathbf{Z}_t \,,$$

where

$$\mathbf{Z}_t := \{(x, y) \in \mathbf{X}_{t-1} \times \mathbf{X}_t : \; (x(\omega), y(\omega)) \in Z_t(\omega) \text{ a.s.}\}.$$

$Z_t(\omega)$ cone (*transition cone*) depending $\mathbf{F}_t$-measurably on ω.

$(x_{t-1}, x_t) \in \mathbf{Z}_t \Leftrightarrow$ portfolio x_{t-1} can be transformed to x_t under the *self-financing constraint* (with *transaction costs*).

Self-financing trading strategies: paths in this system.

An example of the transition cone $Z_t(\omega)$ in a financial market model with transaction costs

There are n assets. Given:

$$\underline{S}_t^i(\omega) < \overline{S}_t^i(\omega),\ i = 1, \ldots, n,$$

the asset i's **bid** and **ask** prices, respectively. (You get $\underline{S}_t^i(\omega)$ when you *sell* and pay $\overline{S}_t^i(\omega)$ when you *buy*.)

The cone $Z_t(\omega)$ consists of (a, b) satisfying

$$\sum_{i=1}^{n} \overline{S}_t^i (b^i - a^i)_+ \leq \sum_{i=1}^{n} \underline{S}_t^i (a^i - b^i)_+ \ ,$$

where $r_+ := \max\{r, 0\}$.

According to the definition of $Z_t(\omega)$, *asset purchases are made only at the expense of sales* of available assets (under transaction costs).

Financial applications of the von Neumann-Gale theory:

I. E., M.A.H. Dempster and M.I. Taksar, Asset pricing and hedging in financial markets with transaction costs: An approach based on the von Neumann-Gale model, 2006, **Annals of Finance**, v. 2, 327-355.

I. E., W. Bahsoun and M.I. Taksar, Growth-optimal investments and numeraire portfolios under transaction costs. **Handbook of the Fundamentals of Financial Decision Making**. (L.C. MacLean and W.T. Ziemba, eds.), 2013.

Dual paths

Sequences $p_0, p_1, \ldots$ ($p_t \geq 0$, p_t is $\mathbf{F}_t$-measurable) satisfying

$$Ep_t y \leq Ep_{t-1} x, \quad (x,y) \in \mathbf{Z}_t \ ,$$

are called **dual paths**.

Note that if (p_t) is a dual path, then for any path (y_t)

$$Ep_0 y_0 \geq Ep_1 y_1 \geq \ldots \geq Ep_t y_t \geq \ldots$$

In financial models, dual paths represent *market-consistent price systems*. In the models of frictionless markets, they can be expressed through densities of equivalent martingale measures.

If (x_t) is a path, (p_t) is a dual path and

$$p_t x_t = 1 \text{ (a.s.)}$$

then (p_t) is said to **support** (x_t).

Assumption (duality L_∞, L_1): all primal variables x_t are essentially bounded; all dual variables p_t are integrable.

The primary focus is on the analysis of *rapid paths*.

Rapid paths

A path (x_t) is called **rapid** if there exists a dual path (p_t) that supports (x_t) (i.e. $p_t x_t = 1$ a.s.).

Thus **rapid paths are those which are supported by dual paths**.

Why "rapid"?

Rapid paths

The term "*rapid*" is motivated by the fact that rapid paths grow **faster** than others.

Proposition. Let (x_t) be a path. Let $p_0, p_1, \ldots [p_t \in \mathbf{X}_t]$ be a sequence of random vectors such that

$$p_t x_t = 1 \text{ a.s.}$$

Then the following conditions are equivalent:

(i) (p_t) is a dual path, and so (x_t) **is a rapid path**;

(ii) (x_{t-1}, x_t) **maximizes the expected growth rate**:

$$E\frac{p_t y}{p_{t-1} x} \leq E\frac{p_t x_t}{p_{t-1} x_{t-1}} \ (= 1), \ (x, y) \in \mathbf{Z}_t;$$

(iii) (x_{t-1}, x_t) **maximizes the expected logarithm of the growth rate**:

$$E \ln \frac{p_t y}{p_{t-1} x} \leq E \ln \frac{p_t x_t}{p_{t-1} x_{t-1}} = 0, \ (x, y) \in \mathbf{Z}_t;$$

(iv) (x_{t-1}, x_t) **maximizes the ratio** $Ep_t y / Ep_{t-1} x$:

$$\frac{Ep_t y}{Ep_{t-1} x} \leq \frac{Ep_t x_t}{Ep_{t-1} x_{t-1}} = 1, \ (x, y) \in \mathbf{Z}_t.$$

Asymptotic growth-optimality of rapid paths

The most important property of infinite rapid paths (which does not depend on (p_t)) is their **asymptotic growth-optimality**.

For a vector $b = (b^1, \ldots, b^n)$, put $|b| = |b^1| + \ldots + |b^n|$.

Under a general assumption on the cones $Z_t(\omega)$ (see (**A3**) below), any rapid path is (**asymptotically**) **growth-optimal**: for any other path $y_0, y_1, \ldots$, we have

$$\sup \frac{|y_t|}{|x_t|} < \infty \text{ a.s.}$$

Remark. The above property remains valid if $|\cdot|$ is replaced by any (possibly random) function $\psi_t(\cdot)$, where $c|a| \leq \psi_t(a) \leq C|a|$, where $0 < c < C$ are some random variables.

For example, $\psi_t(a)$ can be **the value**

$$\psi_t(a) = S_t a$$

of the portfolio a **in some price system** S_t $(0 < c < S_t^i < C)$.

In the financial interpretation, asymptotic optimality means that *no investment strategy* $y_0, y_1, \ldots$ *can yield asymptotically faster growth of wealth than* $x_0, x_1, \ldots$

Existence of rapid paths

Assumptions on the transition cones $Z_t(\omega)$:

(**A1**) There exists a constant M such that $|b|\leq M|a|$ for all $(a,b) \in Z_t(\omega)$.

(**A2**) There exists a constant $\gamma > 0$ such that $(e,\gamma e) \in Z_t(\omega)$, where $e = (1,1,\dots,1)$.

(**A3**) There exists an integer $l \geq 1$ such that for every $t \geq 0$ and $i = 1,\dots,n$ there is a path $y_{t,i},\dots,y_{t+l,i}$ satisfying

$$y_{t,i} = e_i,\dots,y_{t+l,i} \geq \kappa e,\ \kappa > 0,$$

where $e_i = (0,0,\dots,1,\dots,0)$ (ith coordinate is 1).

Theorem. (**A1**-**A2**) *For each $x_0(\omega) \geq \alpha e$ ($\alpha > 0$) and each N there exists a finite rapid path of length N with initial state x_0.*

(**A1**-**A3**) *For each $x_0(\omega) \geq \alpha e$ ($\alpha > 0$) there exists an infinite rapid path with initial state x_0.*

Finite rapid paths are solutions to optimization problems of the form

$$E \ln \psi(x_N) \to \max$$

(ψ concave, monotone, homogeneous).

I. E. and S. D. Flåm, Rapid growth paths in multivalued dynamical systems generated by homogeneous convex stochastic operators, 1998, **Set-Valued Analysis**, v. 6, 61-82.

Infinite paths (general assumptions): W. Bahsoun, I. E. and M. Taksar, **Stochastics**, 2008, v. 80, 129-142.

Earlier results (under stronger assumptions): I. E. and M. Taksar, **Stochastics and Dynamics**, 2001, v. 1, 493-509. Based on

I. E., S.V. Anoulova and V.M. Gundlach, Turnpike theorems for positive multivalued stochastic operators, 2000, **Adv**. **Math**. **Econ**., v. 2, 1-20.

Autonomous systems

Autonomous random dynamical systems serve as a framework for *stationary stochastic models*.

In an autonomous von Neumann-Gale system, we are given additionally an automorphism of the probability space $(\Omega, \mathbf{F}, P)$ (*time shift*) — a *measure preserving* one-to-one transformation $T : \Omega \to \Omega$ such that

(a) the filtration $\ldots \subseteq \mathbf{F}_{-1} \subseteq \mathbf{F}_0 \subseteq \mathbf{F}_1 \subseteq \ldots \subseteq \mathbf{F}_t \subseteq \ldots$ (defined here for all $t = 0, \pm 1, \ldots$) is invariant

$$T^{-1}\mathbf{F}_t = \mathbf{F}_{t+1},$$

(b) for each t, we have $Z_t(T\omega) = Z_{t+1}(\omega)$.

Condition (b) means that the cone-valued process $Z_t(\omega)$ is *stationary* (stationarity is understood in terms of *ergodic theory* of dynamical systems).

Definition. A stochastic process (point- or set-valued) $\xi_0(\omega), \xi_1(\omega), \ldots$ is called **stationary** if $\xi_{t+1}(\omega) = \xi_t(T\omega)$ (equivalently, $\xi_t(\omega) = \xi_0(T^t\omega)$).

Balanced paths

In autonomous systems, a central role is played by **balanced paths**, i.e. paths of the form

$$x_t := \lambda(T\omega)\lambda(T^2\omega)\ldots\lambda(T^t\omega)x(T^t\omega),$$

where

$$\lambda(\omega) > 0$$

is an $\mathbf{F}_0$-measurable scalar function and

$$x(\omega) \geq 0$$

is an $\mathbf{F}_0$-measurable vector function satisfying

$$|x(\omega)| = 1.$$

A balanced path *grows with stationary proportions* $x(T^t\omega)$ and *at a stationary rate* $\lambda(T^t\omega)$.

In the deterministic case:

$$x_t = \lambda^t x.$$

Von Neumann path

A balanced path maximizing

$$E \log \lambda(\omega)$$

among all balanced paths is called a **von Neumann path**.

The main results for autonomous systems (**stationary models**)

Assume (**A1**) - (**A3**).

Theorem. *A von Neumann path exists and is rapid.*

A triplet of functions

$$\lambda(\bullet), x(\bullet), p(\bullet)$$

is called a **von Neumann equilibrium** if

$$x_t := \lambda(T\omega)\lambda(T^2\omega)\ldots\lambda(T^t\omega)x(T^t\omega),$$

is a von Neumann path and

$$p_t := \frac{p(T^t\omega)}{\lambda(T\omega)\lambda(T^2\omega)\ldots\lambda(T^t\omega)}$$

is a dual path supporting it.

Theorem. *A von Neumann equilibrium exists.*

Dynkin's problem. The above results give the positive answer to the existence problem for a stochastic von Neumann equilibrium posed by Eugene Dynkin in the early 1970s. In the deterministic case: von Neumann (1937), Gale (1956).

The existence theorem for a von Neumann equilibrium was obtained in the paper:

I. E. and K. Schenk-Hoppé, Stochastic equilibria in von Neumann-Gale dynamical systems, 2008, **Transactions of the American Mathematical Society**, 360, 3345-3364.

The paper relied substantially on the previous work:

I. E. and K. Schenk-Hoppé, Pure and randomized equilibria in the stochastic von Neumann-Gale model, 2007, **Journal of Mathematical Economics**, 43, 871-887.

L. Arnold, I. E. and V. M. Gundlach, Convex-valued random dynamical systems: A variational principle for equilibrium states, 1999, **Random Operators and Stoch**. **Equations**, 7, 23-38.

Existence of a v.N. equilibrium: the strategy of the proof

The proof is based on the idea of ”elimination of randomization” (Dvoretzky, Wald and Wolfowitz, 1950).

First an appropriate extension of the original dynamical system is constructed, using an “additional source of randomness”.

Then, based on some subtle properties of convexity, the randomization is eliminated and the existence of a von Neumann equilibrium in the original system is established.

Stationary model.

Fix some time period $(t-1,t)$, say $(0,1)$ (by stationarity, it does not matter which one).

Put

$$\gamma(\omega,a,b) := \max\{r \geq 0 : \ (a,rb) \in Z_1(\omega)\}$$

for a,b in the unit simplex

$$\Delta := \{a \geq 0 : |a| = 1\}$$

(**characteristic function** of the cone $Z_1(\omega)$).

Proposition. *A von Neumann path exists if and only if the variational problem*

$$E \ln \gamma(\omega, y(\omega), y(T\omega)) \ \to \max,$$

$$y(\omega) \in \Delta,$$

$y(\omega)$ is $\mathbf{F}_0$*-measurable,*

has a solution.

Randomized stationary mass transfer problem.

$$E \int_{\Delta\times\Delta} \ln \gamma(\omega, a, b)\mu(\omega, da, db) \to \max,$$

$\mu(\omega, \cdot, \cdot)$ is a random measure on $\Delta \times \Delta$,

$\mu(\omega, \cdot, \Delta)$ is $\mathbf{F}_0$-measurable,

$$\mu(\omega, \Delta, C) = \mu(T\omega, C, \Delta) \ \forall\ C \subseteq \Delta.$$

Proving that an optimal μ exists (easy).

Elimination of randomization: proving that there is an optimal μ the form

$$\mu(\omega, da, db) = \delta_{y(\omega)}(da) \otimes \delta_{y(T\omega)}(db)$$

for some $\mathbf{F}_0$-measurable $y(\omega)$.

Ergodic theory $+$ convex analysis in spaces of measurable functions.

Deterministic counterpart: **the Romanovskii (1967) version of the mass transfer problem**.

$$\int_{\Delta\times\Delta} K(a,b)\mu(da,db) \to \max$$

$$\mu(\Delta\times\Delta) = 1;$$

$$\mu(\Delta,A) = \mu(A,\Delta) \text{ for all } A\subseteq\Delta.$$

Key question: does there exist a solution of the form

$$\mu(da,db) = \delta_y(da)\otimes\delta_y(db)$$

(for some y)? If yes, then clearly, y maximizes $K(x,x)$.

Characteristic function of a cone. Let Z be the v.N.-G. cone. Put

$$\gamma(a,b) := \max\{r\geq 0 : \ (a,rb)\in Z\}.$$

Define

$$K(a,b) = \ln\gamma(a,b).$$

The existence of a v. N. path ⇔ positive answer to the above question.

An important special case: **Stochastic Perron-Frobenius theory**

Given: probability space $(\Omega, \mathbf{F}, P)$,

filtration $\ldots \subseteq \mathbf{F}_0 \subseteq \mathbf{F}_1 \subseteq \ldots \subseteq \mathbf{F}_t \subseteq \ldots \mathbf{F}$

automorphism $T : \Omega \to \Omega$:

$$TP = T^{-1}P = P, \;\; \mathbf{F}_{t+1} = T^{-1}\mathbf{F}_t,$$

random matrix $0 \leq A(\omega) : R^n \to R^n$, $\mathbf{F}_1$-measurable.

Von Neumann-Gale cones of the form:

$$Z_t(\omega) = \{(a, b) : \;\; b = A(T^{t-1}\omega)a\}.$$

Random Perron-Frobenius eigenvalue and eigenvector

Suppose $A(\omega)A(T\omega)\ldots A(T^l\omega) > 0$ for some $l = l(\omega)$.

Theorem. *There is a random scalar* $\lambda(\omega) > 0$ *(*$\mathbf{F}_1$*-measurable) and a random vector* $x(\omega) > 0$ *(*$\mathbf{F}_0$*-measurable) such that*

$$\lambda(\omega)x(T\omega) = A(\omega)x(\omega), \ |x(\omega)| = 1.$$

The pair $(\lambda, x) \geq 0$ *satisfying the above two equations is unique.*

Von Neumann equilibrium in this system: (x, λ, p),

λ and x are the random P.-F. eigenvalue and eigenvector for $A(\bullet)$ and T, and p can be expressed through the random P.-F. eigenvector of $A^*(\bullet)$ and T^{-1}.

Nonlinear generalization

Analogous results hold a class of nonlinear random mappings $A(\omega, x) : \Omega \times R_+^n \to R_+^n$.

The main assumptions: **homogeneity** and **monotonicity** of the mappings.

For two vectors $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$, we write $x \prec y$ if $x \le y$ and $x \neq y$.

A mapping $A : R_+^n \to R_+^n$ is called *monotone* if $A(x) \le A(y)$ for any vectors $x, y \in R_+^n$ satisfying $x \le y$.

It is called *completely monotone* if it preserves each of the relations $x \le y$, $x \prec y$ and $x < y$ between two vectors $x, y \in R_+^n$ (clearly, if A preserves the second relation, it preserves the first).

A mapping A is termed *strictly monotone* if the relation $x \prec y$ implies $A(x) < A(y)$.

Let $A(x)$ be linear, i.e., defined by a non-negative matrix A. Then:

A preserves the relation $\succ$ $\Leftrightarrow$ A does not have zero columns;

A preserves the relation $>$ $\Leftrightarrow$ A does not have zero rows;

A is completely monotone $\Leftrightarrow$ A has no zero rows and columns.

A is strictly monotone $\Leftrightarrow$ $A > 0$.

Consider the (nonlinear) *cocycle*

$$C(t,\omega) = A(T^{t-1}\omega)A(T^{t-2}\omega)\ldots A(\omega),\ \ t = 1,2,\ldots,$$

[We write for convenience $A(\omega)x = A(\omega,x)$, and the product means the composition of maps.]

Assumptions:

The mapping $A(\omega,x)$ is measurable in ω for each x and it is *completely monotone*, *homogeneous* and *continuous* in x for each ω. For almost all $\omega \in \Omega$, there is a natural number l (depending on ω) such that the mapping $C(l,\omega)$ is *strictly monotone*.

Theorem. *There exists a measurable vector function* $x(\omega) > 0$ *and a measurable scalar function* $\alpha(\omega) > 0$ *such that*

$$\alpha(\omega)x(T\omega) = A(\omega)x(\omega),\ \ |x(\omega)| = 1 \quad \textit{(a.s.)}.$$

This pair of functions $(\alpha(\cdot),x(\cdot)) \ \geq 0$ *is unique up to the a.s. equivalence with respect to* P.

Hilbert-Birkhoff metric

Y the (relative) interior of the unit simplex:

$$Y := \{y > 0 : |y| = 1\}.$$

For $x, y \in Y$,

$$\rho(x,y) = \ln[\max_i \frac{x_i}{y_i} \cdot \max_j \frac{y_j}{x_j}].$$

This is a complete metric on Y (the *Hilbert-Birkhoff metric*).

A key role in the proofs is played by the following fact. Let A be a mapping $R_+^n \to R_+^n$ such that $A(x) \neq 0$ for $x \in Y$. Define

$$f(x) = \frac{A(x)}{|A(x)|}, \ x \in Y.$$

(Recall that $Y := \{y > 0 : |y| = 1\}$.)

Theorem. *If $A(x)$ is homogeneous and strictly monotone, then $f(x)$ is contracting on Y in the H-B metric ρ, i.e.*

$$\rho(f(x), f(y)) < \rho(x,y)$$

for $x, y \in Y$ with $x \neq y$.

References

Linear case:

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I.E. and S.A. Pirogov, A stochastic contraction principle, 2007, **Random Operators and Stochastic Equations**, v. 15, 155-162.

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